An Unobserved Components Model of Total Factor Productivity and the Relative Price of Investment

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Abstract

This paper applies the common stochastic trends representation approach to the time series of total factor productivity and the relative price of investment to investigate the modelling of neutral technology and investment-specific technology and its econometric ramifications on the analysis of general equilibrium model. The permanent and transitory movements in both series are estimated efficiently via MCMC methods using band matrix algorithms. The results indicate that total factor productivity and the relative price of investment are, each, well-represented by a differenced first-order autoregressive process. In addition, their time series share a common trend component that we interpret as reflecting changes in General Purpose Technology. These results are consistent with studies that suggest that (1) the traditional view of assuming that neutral technology and investment-specific technology follow independent processes is not supported by the features of the time series and (2) improper specification of secular trends may distort estimation and inference. Notably, the findings provide some guidance to minimize the effect of idiosyncratic and common trend misspecifications on the analysis of impulse dynamics and propagation mechanisms in macroeconomic models.

Keywords: Business cycles, Investment-specific technological change, Total Factor Productivity, Unobserved Components Model. *JEL codes*: E22, E32, C32.

1 Introduction

Real Business Cycle (RBC) models and their modern versions in Dynamic Stochastic General Equilibrium (DSGE) models have enjoyed a great deal of success since they were first introduced to the literature by Kydland and Prescott (1982). These models are the standard framework used by academics and policymakers to understand economic fluctuations and analyze the effects of monetary and fiscal policies on the macroeconomy. They typically feature rational optimizing agents and various sources of random disturbances to preferences, technology, government purchases, monetary policy rules, and/or international trade. Two notable disturbances, that are now considered standard, are shocks to neutral technology and shocks to investment-specific technology. Neutral technology shocks make both labor and existing capital more productive. On the other hand, investment-specific technology shocks have no impact on the productivity of old capital goods, but they make new capital goods more productive and less expensive.

Nonetheless, the modelling of these two types of technology in many DSGE models often varies greatly from a stationary process to an integrated smooth trend process, with each choice having different implications for the variations in economic activity and the analysis of macro variables. In this paper, we revisit the question of the specification of neutral and investment-specific technology via the lens of an unobserved components (UC) model of the total factor productivity (TFP) and the relative price of investment (RPI). Specifically, the approach allows us to decompose a series into two unobserved components: one unobserved component that reflects permanent or trend movements in the series, and the other that captures transitory movements in the series. In addition, based on the stock of evidence from Schmitt-Grohé and Uribè (2011), Benati (2013), Basu et al. (2013), Chen and Wemy (2015), and other studies that appear to indicate that the two series are related in the long-run, we specify that RPI and TFP share a common unobserved trend component. We label this component as General Purpose Technology (GPT), and we argue that it reflects spillover effects from innovations in information and communication technologies to aggregate productivity. As demonstrated in other studies, we contend that such an analysis of the time series properties of TFP and RPI must be the foundation for the choice of the stochastic processes of neutral technology and investment-specific technology in DSGE models.

The UC framework offers ample flexibility and greater benefits. First, our framework nests all competing theories of the univariate and bivariate properties of RPI and TFP. In fact, the framework incorporates Schmitt-Grohé and Uribè (2011)'s result of co-integration and Benati (2013)'s findings of the long-run positive comovement as special cases. In that sense, we are able to evaluate the validity of all the proposed specifications of neutral technology and investment-specific technology in the DSGE literature. Furthermore, the UC structure yields a quantitative estimate, in addition to the qualitative measure in Benati (2013), of potential scale differences between the trends of RPI and TFP. Last, but not least, it is grounded on economic theory. In particular, we demonstrate that our UC framework may be derived from the neoclassical growth model used by studies in the growth accounting literature, e.g Greenwood et al. (1997), Oulton (2007), and Greenwood and Krusell (2007), to investigate the contribution of embodiment in the growth of aggregate productivity. As such, we may easily interpret the idiosyncratic unobserved component of RPI and TFP, and their potential interaction from the lenses of a well-defined economic structure.

Using the time series of the logarithm of (the inverse of) RPI and TFP from 1959.II to 2019.II in the United States, we estimate our UC model through Markov chain Monte Carlo (MCMC) methods developed by Chan and Jeliazkov (2009) and Grant and Chan (2017). A novel feature of this approach is that it builds upon the band and sparse matrix algorithms for state space models, which are shown to be more efficient than the conventional Kalman filter-based algorithms. Two points emerge from our estimation results.

First, our findings indicate that the idiosyncratic trend component in RPI and TFP is better captured by a differenced first-order autoregressive, ARIMA(1,1,0), process; a result which suggests that neutral technology and investment-specific technology should each be modelled as following an ARIMA(1,1,0) process. While several papers like Justiniano et al. (2011), Schmitt-Grohé and Uribè (2011), and Kaihatsu and Kurozumi (2014) adopt an ARIMA(1,1,0) specification, other notable studies as Smets and Wouters (2007) and Fisher (2006) impose either a trend stationary ARMA(1,0) process or an integrated ARIMA(0,1,0) process, respectively. Through our exploration of the literature, we find that this disconnected practice is complicated by two important facts. The first fact is that it is not easy to establish whether highly persistent macro data are trend stationary or difference stationary in finite sample. Since the models are expected to fit the data along this dimension, researchers typically take a stand on the specification of the trend in DSGE models by arbitrarily building into these models stationary or non-stationary components of neutral technology and investment-specific technology. Then, the series are transformed accordingly in the same fashion prior to the estimation of the parameters of the models. However, several studies demonstrate that this approach may create various issues such as generating spurious cycles and correlations in the filtered series, and/or altering the persistence and the volatility of the original series. The second fact, which is closely related to the first, is that DSGE models must rely on impulse dynamics to match the periodicity of output as such models have weak amplification mechanisms. In DSGE models, periodicity is typically measured by the autocorrelation function of output. Many studies document that output growth is positively correlated over short horizons. Consequently, if technology follows a differenced first-order autoregressive, ARIMA(1,1,0), process, the model will approximately mimic the dynamics of output growth in the data. Therefore, our results serve to reduce idiosyncratic trend misspecification which may lead to erroneous conclusions about the dynamics of macroeconomic variables.

The second point is that while the idiosyncratic trend component in each (the inverse of RPI and TFP) is not common to both series, it appears that the variables share a common stochastic trend component which captures a positive long-run covariation between the series. We argue that the common stochastic trend component is a reflection of General Purpose Technological (GPT) progress from innovations in information and communications technologies. In fact, using industry-level and aggregate-level data, several studies, such as Cummins and Violante (2002), Basu et al. (2004), and Jorgenson et al. (2007), document that improvements in information communication technologies contributed to productivity growth in the 1990s and the 2000s in essentially every industry in the United States. Through our results, we are also able to confirm that changes in the trend of RPI have lasting impact on the long-run developments of TFP. In addition, errors associated with the misspecification of common trend between neutral technology and investment-specific technology may bias conclusions about the source of business cycle fluctuations in macroeconomic variables. For instance, Schmitt-Grohé and Uribè (2011) shows that when neutral technology and investment-specific technology are co-integrated, then the shocks to the common stochastic trend become the major source of fluctuations of output, investment and hours. This challenges results in several studies which demonstrate that exogenous disturbances in either neutral technology or investment-specific technology, may individually, account for the majority of the business cycle variability in economic variables. Therefore, our results suggest that researchers might need to modify DSGE models to consider the possibility of the existence of this long-run relationship observed in the data and its potential effect on fluctuations.

2 The Model

Fundamentally, there are multiple potential representations of the relationship between the trend components and the transitory components in RPI and TFP. We adopt the Unobserved Component (UC) approach which stipulates that RPI and TFP can each be represented as the sum of a permanent component, an idiosyncratic component, and a transitory component in the following fashion:

$$z_t = \tau_t + c_{z,t}, \tag{1}$$

$$x_t = \gamma\tau_t + \tau_{x,t} + c_{x,t}, \tag{2}$$

where z_t is the logarithm of RPI, x_t is the logarithm of TFP, τ_t is the common trend component in RPI and TFP, $\tau_{x,t}$ is the idiosyncratic trend component in TFP, and $c_{z,t}$ and $c_{x,t}$ are the corresponding idiosyncratic transitory components. The parameter γ captures the relationship between the trends in RPI and TFP.

The first differences of the trend components are modelled as following stationary processes,

$$\Delta\tau_t = (1-\varphi_\mu)\zeta_1 1(t < T_B) + (1-\varphi_\mu)\zeta_2 1(t \geq T_B) + \varphi_\mu \Delta\tau_{t-1} + \eta_t, \tag{3}$$

$$\Delta\tau_{x,t} = (1-\varphi_{\mu_x})\zeta_{x,1} 1(t < T_B) + (1-\varphi_{\mu_x})\zeta_{x,2} 1(t \geq t_0) + \varphi_{\mu_x} \Delta\tau_{x,t-1} + \eta_{x,t}, \tag{4}$$

where $\eta_t \sim \mathcal{N}(0,\sigma^2_\eta)$ and $\eta_{x,t} \sim \mathcal{N}(0,\sigma^2_{\eta_x})$ are independent of each other at all leads and lags, $1(A)$ is the indicator function for the event A, and T_B is the index corresponding to the time of the break at $1982.I$.

Finally, following Morley et al. (2003) and Grant and Chan (2017), the transitory components are assumed to follow AR(2) processes:

$$c_{z,t} = \phi_{z,1} c_{z,t} + \phi_{z,2} c_{z,t} + \varepsilon_{z,t}, \tag{5}$$

$$c_{x,t} = \phi_{x,1} c_{x,t} + \phi_{x,2} c_{x,t} + \varepsilon_{x,t}, \tag{6}$$

where $\epsilon_{z,t} \sim \mathcal{N}(0, \sigma_z^2)$, $\epsilon_{x,t} \sim \mathcal{N}(0, \sigma_x^2)$, and $corr(\varepsilon_{z,t}, \varepsilon_{x,t}) = 0$.[1]

The presence of the common stochastic trend component in RPI and TFP captures the argument that innovations in information technologies are General Purpose Technology (GPT), and they have been the main driver of the trend in RPI and a major source of the growth in productivity in the United States. Simply put, GPT can be defined as a new method that leads to fundamental changes in the production process of industries using it, and it is important enough to have a protracted aggregate impact on the economy. As discussed extensively in Jovanovic and Rousseau (2005), electrification and information technology (IT) are probably the most recent GPTs so far. In fact, using industry-level data, Cummins and Violante (2002) and Basu et al. (2004) find that improvements in information communication technologies contributed to productivity growth in the 1990s in essentially every industry in the United States. Moreover, Jorgenson et al. (2007) show that much of the TFP gain in the United States in the 2000s originated in industries that are the most intensive users of information technology. Specifically, the authors look at the contribution to the growth rate of value added and aggregate TFP in the United States in 85 industries. They find that the four IT-producing industries (computer and office equipment, communication equipment, electronic components, and computer services) accounted for nearly all of the acceleration of aggregate TFP in 1995-2000. Furthermore, IT-using industries, which engaged in great IT-investment in the period 1995-2000, picked up the momentum and contributed almost half of the aggregate acceleration in 2000-2005. Overall, the authors assert that IT-related industries made significant contributions to the growth rate of TFP in the period 1960-2005. Similarly, Gordon (1990) and Cummins and Violante (2002) have argued that technological progress in areas such as equipment and software

[1] We note that economic theory does not rule out the presence of correlation in transitory movements of two or more time series. In fact, allowing $corr(\varepsilon_{z,t}, \varepsilon_{x,t}) \neq 0$ would be a valuable exercise as it would allow us to capture the possibility of short to medium horizon business cycle movements between RPI and TFP, and potentially provide some insights into the cyclical behavior of RPI and/or TFP. However, for our purpose to better understand the idiosyncratic and joint specification of neutral technology and investment-specific technology, we focus on the modelling of the permanent components in RPI and TFP and leave this potential extension of the transitory components to future research.

have contributed to a faster rate of decline in RPI; a fact that has also been documented in Fisher (2006) and Justiniano et al. (2011).

A complementary interpretation of the framework originates from the growth accounting literature associated with the relative importance of embodiment in the growth of technology. In particular, Greenwood et al. (1997), Greenwood and Krusell (2007) and Oulton (2007) show that the non-stationary component in TFP is a combination of the trend in neutral technology and the trend in investment specific technology. In that case, we may interpret the common component, τ_t, as the trend in investment-specific technology, $\tau_{x,t}$ as the trend in neutral technology, and the parameter γ as the current price share of investment in the value of output. In Appendix A.1, we show that our UC framework may be derived from a simple neoclassical growth model.

Furthermore, this UC framework offers more flexibility as it nests all the univariate and bivariate specifications of neutral technology and investment-specific technology in the literature. Specifically, let us consider the following cases:

1. If $\gamma = 0$, then trends in RPI and TFP are independent of each other, and this amounts to the specifications adopted in Justiniano et al. (2011) and Kaihatsu and Kurozumi (2014).

2. If $\gamma = 0$, $\varphi_\mu = 0$ and $\varphi_{\mu_x} = 0$, then the trend components follow a random walk plus drift, and the resulting specification is equivalent to the assumptions found in Fisher (2006).

3. If we pre-multiply equation (1) by γ, and subtract the result from equation (2), we obtain

$$x_t - \gamma z_t = \tau_{x,t} + c_{xz,t} \tag{7}$$

   where $c_{xz,t} = c_{x,t} - \gamma c_{z,t}$.

   (a) If $\gamma \neq 0$, $\varphi_{\mu_x} = 0$ and $\sigma_{\eta_x} = 0$, then RPI and TFP are co-integrated, with co-integrating vector $(-\gamma, 1)$, as argued in Schmitt-Grohé and Uribè (2011).

   (b) If, in addition, $\zeta_x \neq 0$, then $\tau_{x,t}$ is a linear deterministic trend, and RPI and TFP are co-integrated around a linear deterministic trend.

   (c) If, otherwise, $\zeta_x = 0$, the RPI and TFP are co-integrated around a constant term.

4. If $\gamma > 0$ ($\gamma < 0$), then the common trend component has a positive effect on both RPI and

TFP (a positive effect on RPI and a negative effect on TFP), which would imply a positive (negative) covariation between the two series. This is essentially the argument in Benati (2013).

5. If $\varphi_\mu = 1$ and $\varphi_{\mu_x} = 1$. This gives rise to the following smooth evolving processes for the trend:

$$\Delta\tau_t = \zeta_1 1(t < T_B) + \zeta_2 1(t \geq T_B) + \Delta\tau_{t-1} + \eta_t, \tag{8}$$

$$\Delta\tau_{x,t} = \zeta_{x,1} 1(t < T_B) + \zeta_{x,2} 1(t \geq t_0) + \Delta\tau_{x,t-1} + \eta_{x,t}. \tag{9}$$

Through a model comparison exercise, we are able to properly assess the validity of each of these competing assumptions and their implications for estimation and inference in order to shed light on the appropriate representation between neutral technology and investment-specific technology.

3 Bayesian Estimation

In this section we provide the details of the priors and outline the Bayesian estimation of the unobserved components model in (1)–(6). In particular, we highlight how the model can be estimated efficiently using band matrix algorithms instead of conventional Kalman filter based methods.

We assume proper but relatively noninformative priors for the model parameters γ, $\boldsymbol{\phi} = (\phi_{z,1}, \phi_{z,2}, \phi_{x,1}, \phi_{x,2})'$, $\boldsymbol{\varphi} = (\varphi_\mu, \varphi_{\mu_x})'$, $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_{x,1}, \zeta_{x,2})'$, $\boldsymbol{\sigma}^2 = (\sigma^2_\eta, \sigma^2_{\eta_x}, \sigma^2_z, \sigma^2_x)'$ and $\boldsymbol{\tau}_0$. In particular, we adopt a normal prior for γ: $\gamma \sim \mathcal{N}(\gamma_0, V_\gamma)$ with $\gamma_0 = 0$ and $V_\gamma = 1$. These values imply a weakly informative prior centered at 0. Moreover, we assume independent priors for $\boldsymbol{\phi}$, $\boldsymbol{\varphi}$, $\boldsymbol{\zeta}$ and $\boldsymbol{\tau}_0$:

$$\boldsymbol{\phi} \sim \mathcal{N}(\boldsymbol{\phi}_0, \mathbf{V}_{\boldsymbol{\phi}})1(\boldsymbol{\phi} \in \mathbf{R}),\ \boldsymbol{\varphi} \sim \mathcal{N}(\boldsymbol{\varphi}_0, \mathbf{V}_{\boldsymbol{\varphi}})1(\boldsymbol{\varphi} \in \mathbf{R}),\ \boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{\zeta}_0, \mathbf{V}_{\boldsymbol{\zeta}}),\ \boldsymbol{\tau}_0 \sim \mathcal{N}(\boldsymbol{\tau}_{00}, \mathbf{V}_{\boldsymbol{\tau}_0}),$$

where $\mathbf{R}$ denotes the stationarity region. The prior on the AR coefficients $\boldsymbol{\phi}$ affects how persistent the cyclical components are. We assume relatively large prior variances, $\mathbf{V}_{\boldsymbol{\phi}} = \mathbf{I}_4$, so that *a priori* $\boldsymbol{\phi}$ can take on a wide range of values. The prior mean is assumed to be $\boldsymbol{\phi}_0 = (1.3, -0.7, 1.3, -0.7)'$, which implies that each of the two AR(2) processes has two complex roots, and they are relatively persistent. Similarly, for the prior on $\boldsymbol{\varphi}$, we set $\mathbf{V}_{\boldsymbol{\varphi}} = \mathbf{I}_2$ with prior mean $\boldsymbol{\varphi}_0 = (0.9, 0.9)'$, which implies that the two AR(1) processes are fairly persistent. Next, we assume that the priors on σ_z^2 and σ_x^2 are inverse-gamma:

$$\sigma_z^2 \sim \mathcal{IG}(\nu_z, S_z), \quad \sigma_x^2 \sim \mathcal{IG}(\nu_x, S_x).$$

We set $\nu_z = \nu_x = 4$, $S_z = 6 \times 10^{-5}$ and $S_x = 3 \times 10^{-5}$. These values imply prior means of σ_z^2 and σ_x^2 to be, respectively, 2×10^{-5} and $S_x = 10^{-5}$.

For σ_η^2 and $\sigma_{\eta_x}^2$, the error variances in the state equations (3) and (4), we follow the suggestion of Frühwirth-Schnatter and Wagner (2010) to use normal priors centered at 0 on the standard deviations σ_η and σ_{η_x}. Compared to the conventional inverse-gamma prior, a normal prior centered at 0 has the advantage of not distorting the likelihood when the true value of the error variance is close to zero. In our implementation, we use the fact that a normal prior $\sigma_\eta \sim \mathcal{N}(0, V_{\sigma_\eta})$ on the standard deviation implies gamma prior on the error variance σ_η^2: $\sigma_\eta^2 \sim \mathcal{G}(1/2, 1/(2V_{\sigma_\eta}))$, where $\mathcal{G}(a, b)$ denotes the gamma distribution with mean a/b. Similarly, we assume $\sigma_{\eta_x}^2 \sim \mathcal{G}(1/2, 1/(2V_{\sigma_{\eta_x}}))$. We set $V_{\sigma_\eta} = 5 \times 10^{-6}$ and $V_{\sigma_{\eta_x}} = 5 \times 10^{-5}$.

Next, we outline the posterior simulator to estimate the model in (1)–(6) with the priors described above. To that end, let $\boldsymbol{\tau} = (\tau_1, \tau_{x,1}, \tau_2, \tau_{x,2}, \ldots, \tau_T, \tau_{x,T})'$ and $\mathbf{y} = (z_1, x_1, \ldots, z_T, x_T)'$. Then, posterior draws can be obtained by sequentially sampling from the following conditional distributions:

1. $p(\boldsymbol{\tau}, \gamma \,|\, \mathbf{y}, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0) = p(\gamma \,|\, \mathbf{y}, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0) p(\boldsymbol{\tau} \,|\, \mathbf{y}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0)$;

2. $p(\boldsymbol{\phi} \,|\, \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0)$;

3. $p(\boldsymbol{\varphi} \,|\, \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0)$;

4. $p(\boldsymbol{\sigma}^2 \,|\, \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\tau}_0)$;

5. $p(\boldsymbol{\zeta}, \boldsymbol{\tau}_0 \,|\, \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\sigma}^2)$;

We refer the readers to Appendix A.5 for implementation details of the posterior sampler.

4 Empirical Results

In this section we report parameter estimates of the bivariate unobserved components model defined in (1)–(6). The data set consists of the time-series of the logarithm of (the inverse of) RPI and TFP from 1959.II to 2019.II. RPI is computed as the investment deflator divided by the consumption deflator, and it is easily accessible from the Federal Reserve Economic Database (FRED). While the complete description of the deflators along with the accompanying details of the computation of the series are found in DiCecio (2009), it is worth emphasizing that the investment deflator corresponds to the quality-adjusted investment deflator calculated following the approaches in Gordon (1990), Cummins and Violante (2002), and Fisher (2006). On the other hand, we compute TFP based on the aggregate TFP growth, which is measured as the growth rate of the business-sector TFP corrected for capital utilization. The capital utilization adjusted aggregate TFP growth series is produced by Fernald (2014), and is widely regarded as the best available measure of neutral technology.[2]

First, we perform statistical break tests to verify, as has been established in the empirical literature, that RPI experienced a break in its trend around the early 1982's. Following the recommendations in Bai and Perron (2003), we find a break date at 1982.I in the mean of the log difference of RPI as documented in Fisher (2006), Justiniano et al. (2011), and Benati (2013). In addition, a consequence of our specification is that RPI and TFP must share a common structural break. Consequently, we follow the methodology outlined in Qu and Perron (2007) to test whether or not the trends in the series are orthogonal.[3] The estimation results suggest that RPI and TFP might share a common structural break at 1980.I. This break date falls within the confidence interval of the estimated break date between the same two time series, $[1973.I, 1982.III]$, documented in Benati (2013). Therefore, the evidence from structural break tests does not rule out the presence

[2]We use the vintage from September 2019 for our baseline estimation as older versions of the adjusted TFP series are typically less refined. Some studies, such as Kurmann and Sims (2021), have argued that pre- and post-revisions of Fernald's adjusted TFP series, based on a switch in 2014 in detrending methods of hours per worker in the estimation of utilization, matter for the identification of shocks. As a robustness check, we also estimate the model using the May 2014 vintage. Despite the differences in vintage and sample period, the estimation results are mostly similar and all our conclusions remain the same. These additional results are available upon requested. Different vintages of the adjusted TFP series are available on John Fernald's webpage http://www.frbsf.org/economic-research/economists/john-fernald/.

[3]First, we test for the presence of a structural break in TFP following the recommendations in Bai and Perron (2003). We find that the mean of the log difference in TFP exhibits a break at 1968.I; a date that is similar to that documented in Benati (2013) who estimates the break date at 1968.II. Next, following the approach in Qu and Perron (2007), we regress a vector of both, a series of RPI and a series of TFP, on a vector of constant, linear trends and random errors. We use a trimming value $\epsilon = 0.20$ and allow up to 3 breaks.

of a single break at $1982.I$ in the common stochastic trend component of RPI and TFP.

To jumpstart the discussion of the estimation results, we provide a graphical representation of the fit of our model as illustrated by the fitted values of the two series in Figure 1. It is clear from the graph that the bivariate unobserved components model is able to fit both series well with fairly narrow credible intervals.

[Figure 1 about here.]

We report the estimates of model parameters in Table 1 and organize the discussion of our findings around the following two points: *(1)* the within-series relationship, and *(2)* the cross-series relationship. At the same time, we elaborate on the econometric ramifications of these two points on the analysis of Dynamic Stochastic General Equilibrium (DSGE) models.

4.1 The within-series relationship in RPI and TFP

The within-series relationship is concerned with the relative importance of the permanent component and the transitory component in (the inverse of) RPI and TFP. This relationship is captured via the estimated values of *(i)* the drift parameters, ζ and ζ_x, *(ii)* the autoregressive parameters of the permanent components in RPI and TFP, φ_μ and φ_{μ_x}, *(iii)* the autoregressive parameters of the transitory components in RPI and TFP, $\phi_{z,1}$, $\phi_{z,2}$, $\phi_{x,1}$ and $\phi_{x,2}$, *(iv)* and the standard deviations of the permanent and transitory components, σ_η, σ_{η_x}, σ_z, and σ_x.

First, it is evident from Figure 2 that the estimated trend in RPI, τ_t, is strongly trending upward, especially after the break at $1982.I$. In fact, the growth rate of τ_t has more than doubled after $1982.I$: as reported in Table 1, the posterior means of ζ_1 and ζ_2 are 0.004 and 0.009, respectively. This is consistent with the narrative that the decline in the mean growth of RPI has accelerated in the period after $1982.I$, and this acceleration has been facilitated by the rapid price decline in information processing equipment and software. In contrast, the estimated trend $\tau_{x,t}$ shows gradual decline after $1982.I$—its growth rate decreases from 0.001 before the break to -0.003 after the break, although both figures are not statistically different from zero. Overall, it appears that trends in both series are captured completely by the stochastic trend component in RPI, and this piece of evidence provides additional support that GPT might be an important driver of the growth in TFP in the United States.

Moving on to the estimated autoregressive parameters of the permanent component, $\widehat{\varphi}_{\mu} = 0.101$ and $\widehat{\varphi}_{\mu_x} = -0.051$, we note that growths in RPI and TFP do not appear to be as serially correlated as reported in the empirical literature. For instance, Justiniano et al. (2011) report a posterior median value of 0.287 for the investment-specific technological process and 0.163 for neutral technology. With regards to the AR(2) processes that describe the transitory components in RPI and TFP, the estimated autoregressive parameters indicate that these components are relatively more persistent than the growth components of the series.

Furthermore, the estimated values of the variance of the innovations lead to some interesting observations. The variance of the idiosyncratic growth rate in TFP, $\sigma^2_{\eta_x} = 5.17 \times 10^{-5}$, is larger than its counterpart for the transitory component, $\sigma^2_x = 7.76 \times 10^{-6}$. On the other hand, the variance of the growth rate in RPI, $\sigma^2_{\eta} = 5.51 \times 10^{-6}$, is smaller than the variance of its transitory component, $\sigma^2_z = 1.79 \times 10^{-5}$.

Overall, these results seem to indicate that (i) RPI and TFP appear to follow an ARIMA(1,1,0) process, (ii) RPI and TFP growths are only weakly serially correlated, (iii) transitory components in RPI and TFP are relatively more persistent than the growth components on these series, (iv) TFP growth shocks generate more variability than shocks to the growth rate of RPI, and (v) shocks to the transitory component of RPI are more volatile than shocks to the growth rate of RPI.

Now, we discuss the implications of these findings on the analysis of business cycle models. First, information about the process underlying RPI and TFP helps to reduce errors associated with the specification of idiosyncratic trends in DSGE models. Macro variables such as output are highly persistent. To capture this feature, researchers must typically take a stand on the specification of the trend in DSGE models. Since output typically inherits the trend properties of TFP and/or RPI, having accurate information about the trend properties of TFP and/or RPI should allow the researcher to minimize the possibility of trend misspecification which in turn has significant consequences on estimation and inference. Specifically, DSGE models are typically built to explain the cyclical movements in the data. Therefore, preliminary data transformation, in the form of removing secular trend variations, are oftentimes required before the estimation of the structural parameters. One alternative entails arbitrarily building into the model a non-cyclical component of neutral technology (NT hereafter) and/or investment-specific technology (IST hereafter) and filtering the raw data using the model-based specification. Therefore, trend misspecifications may lead to inappropriate filtering approaches and erroneous conclusions. In fact, Singleton (1988) documents that inadequate filtering generally leads to inconsistent estimates of the parameters. Similarly, Canova (2014) recently demonstrates that the posterior distribution of the structural parameters vary greatly with the preliminary transformations (linear detrending, Hodrick and Prescott (HP) filtering, growth rate filtering, and band pass filtering) used to remove secular trends. Consequently, this translates into significant differences in the impulse and propagation of shocks. Furthermore, Cogley and Nason (1995b) show that it is hard to interpret results from filtered data as facts or artifacts about business cycles as filtering may generate spurious cycles in difference stationary and trend-stationary processes. For example, the authors find that a model may exhibit business cycle periodicity and comovement in filtered data even when such phenomena are not present in the data.

Second, information about the process underlying RPI and TFP improves our understanding of the analysis of macro dynamics. Researchers typically analyze the dynamics of output and other macro variables along three dimensions over short horizons: the periodicity of output, comovement of macro variables with respect to output, and the relative volatility of macro variables. Specifically, in macro models, periodicity is usually measured by the autocorrelation function of output, and Cogley and Nason (1993) and Cogley and Nason (1995a) show that the periodicity of output is essentially determined by impulse dynamics. Several studies document that output growth is positively correlated over short horizons, and standard macro models struggle to generate this pattern. Consequently, in models where technology followed an ARIMA (1,1,0) process, output growth would inherit the AR(1) structure of TFP growth, and the model would be able to match the periodicity of output in the data. This point is particularly important as it relates to the responses of macro variables to exogenous shocks. Cogley and Nason (1995a) highlight that macro variables, and output especially, contain a trend-reverting component that has a hump shaped impulse response function. Since the response of output to technology shock, for instance, pretty much matches the response of technology itself, Cogley and Nason (1995a) argue that exogenous shocks must produce this hump shaped pattern for the model to match the facts about autocorrelated output growth. Ultimately, if the goal of the researcher is to match the periodicity of output in a way that the model also matches the time series characteristics of the impulse dynamics, then the choice of the trend is paramount in the specification of the model.

In addition, information about the volatilities and persistence of the processes underlying technology shocks may strengthen our inference on the relative importance of these shocks on fluctuations. As demonstrated in Canova (2014), the estimates of structural parameters depend on nuisance features such as the persistence and the volatility of the shocks, and misspecification of these nuisance features generates biased estimates of impact and persistence coefficients and leads to incorrect conclusions about the relative importance of the shocks.

4.2 The cross-series relationship between (the inverse of) RPI and TFP

We can evaluate the cross-series relationship through the estimated value of the parameter that captures the extent of the relationship between the trends in RPI and TFP, namely, γ.

First, the positive value of $\widehat{\gamma}$ implies a positive long-run co-variation between (the inverse of) RPI and TFP as established in Benati (2013). In addition, the estimated value of the parameter γ, ($\widehat{\gamma} = 0.478$), is significantly different from zero and quite large. This result supplements the qualitative findings by providing a quantitative measure of scale differences in the trends of RPI and TFP. From a econometric point of view, we may interpret $\widehat{\gamma}$ as the elasticity of TFP to IST changes. In that case, we may assert that, for the time period 1959.II to 2019.II considered in this study, a 1 percent change in IST progress leads to a 0.478 percent increase in aggregate TFP. These observations are consistent with the view that permanent changes in RPI, which may reflect improvements in information communication technologies, might be representation of innovations in General Purpose Technology. In fact, using industry-level data , Cummins and Violante (2002) and Basu et al. (2004) find that improvements in information communication technologies contributed to productivity growth in the 1990s in essentially every industry in the United States. Moreover, Jorgenson et al. (2007) show that much of the TFP gain in the United States in the 2000s originated in industries that are the most intensive users of information technology.

In addition, this finding contributes to the debate about the specification of common trends and the source of business cycle fluctuations in DSGE models. In particular, Schmitt-Grohé and Uribè (2011) and Benati (2013) explore the relationship between RPI and TFP through the lenses of statistical tests of units root and co-integration, and the potential implications of such relationship on the role of technology shocks in economic fluctuations. Using quarterly U.S. data over the period from 1948.I to 2006.IV, Schmitt-Grohé and Uribè (2011) find that RPI and TFP contain a non-stationary stochastic component which is common to both series. In other words, TFP and

RPI are co-integrated, which implies that neutral technology and investment-specific technology should be modelled as containing a common stochastic trend. Therefore, Schmitt-Grohé and Uribè (2011) estimate a DSGE model that imposes this result and identify a new source of business cycle fluctuations: shocks to the common stochastic trend in neutral and investment specific productivity. They find that the shocks play a sizable role in driving business cycle fluctuations as they explain three fourth of the variances of output and investment growth and about one third of the predicted variances of consumption growth and hours worked. If such results were validated, they would reshape the common approach of focusing on the importance of either neutral technology or investment-specific technology distinctively.

However, Benati (2013) expresses some doubt about the co-integration results in Schmitt-Grohé and Uribè (2011). He claims that TFP and RPI are most likely not co-integrated, and he traces the origin of this finding of co-integration to the use of an inconsistent criteria for lag order selection in the Johansen procedure. When he uses the SIC or HQ criterion, the Johansen test points to no co-integration. Yet, he establishes that although the two series may not be co-integrated, they may still share a common stochastic non-stationary component. Using an approach proposed by Cochrane and Sbordone (1988) that searches for a statistically significant extent of co-variation between the two series' long-horizon differences, Benati (2013) suggests that the evidence from his analysis points towards a common $I(1)$ component that induces a positive co-variation between TFP and RPI at long horizons. Then, he uses such restrictions in a vector autoregression to identify common RPI and TFP component shocks and finds that the shocks play a sizable role in the fluctuations of TFP, RPI and output: about 30 percent of the variability of RPI and TFP and 28 percent of the variability in output.

First, the results from our analysis complements the investigation in Schmitt-Grohé and Uribè (2011) and Benati (2013). The relatively large estimate of γ validates the notion that neutral technology and investment-specific technology should not be modelled in DSGE models as emanating from orthogonal processes. While the two studies may disagree about the exact nature of the relationship between neutral technology and investment-specific technology, both share the common view that their underlying processes are not independent of each other as shown by the results herein, and disturbances to the process that joins them play a significant role in fluctuations of economic activity. This indicates, obviously, that a proper accounting, based on the time series properties of RPI and TFP, of the joint specification of neutral and investment-specific technology is necessary to address succinctly questions of economic fluctuations as common trend misspecification provide inexact conclusions about the role of shocks in economic fluctuations. For instance, papers that have approached the question about the source of fluctuations from the perspective of general equilibrium models have generated diverging results; partly because of the specification of neutral technology and investment-specific technology. In Justiniano et al. (2010), investment(-specific) shocks account for most of the business cycle variations in output, hours, and investment (at least 50 percent) while neutral technology shocks play a minimal role. On the hand, the findings in Smets and Wouters (2007) flip the script: the authors find that neutral technology (and wage mark up) shocks account for most the output variations while investment(-specific) shocks play no role. Nonetheless, these two studies share the common feature that the processes underlying neutral technology and investment-specific technology are orthogonal to each other. As we discussed above, when Schmitt-Grohé and Uribè (2011) impose a common trend specification between the two types of technology, neither shocks matter for fluctuations: a new shock, the common stochastic trend shocks, emerges as the main source of business cycle fluctuations. Our framework offers a structural way to assimilate the common trend specification in DSGE models and evaluate the effects of such specifications in fluctuations. The typical approach to trend specification in DSGE models is to build a non-cyclical component into the model via unit roots in neutral technology and/or investment-specific technology and filtering the raw data using the model-based transformation. However, this practice produces, in addition to inexact results about sources of fluctuations, counterfactual trend implications because they incorporate balanced growth path restrictions that are to some extent violated in the data. Therefore, Ferroni (2011) and Canova (2014) recommend an alternative one-step estimation approach that allows to specify a reduced-form representation of the trend component, which is ultimately combined to the DSGE model for estimation. Our results about the time series characteristics of RPI and TFP provides some guidance on the specification of such reduced-form models. Fernandez-Villaverde et al. (2016) recommend this alternative approach as one of the most desirable and promising approach to modelling trends in DSDE models. Furthermore, our framework distances itself from statistical tests and their associated issues of lag order and low power as the debate about the long-run relationship between RPI and TFP has thus far depended on the selection of lag order in the Johansen's test. Also, our framework incorporates

Schmitt-Grohé and Uribè (2011)'s result of co-integration and Benati (2013)'s finding of co-variation as special cases.

Finally, our findings about the fact that TFP may contain two non-stationary components contributes to the Vector Autoregression (VAR) literature about the identification of technology shocks and their role in business cycle fluctuations. Specifically, using a bivariate system consisting of the log difference of labor productivity and hours worked, Gali (1999) assesses the role of technology in generating fluctuations by identifying a (neutral) technology shock under the restriction (which could be derived from most standard RBC models) that only such shocks may have a permanent effect on the log level of labor productivity. An (implicit) underlying assumption of this long-run restriction is that the unit root in labor productivity is driven exclusively by (neutral) technology. Fisher (2006) extended Gali (1999)'s framework to show that IST shocks may also have a permanent effect log level of labor productivity, and they play an significant role in generating fluctuations in economic variables. While both authors consider labor productivity as the key variable to their identification process, their argument may easily be applied to the case when we consider TFP instead. In Appendix A.1, we demonstrate that if output is measured in consumption units, only neutral technology affects TFP permanently. However, when output is tabulated according to the Divisia index, both neutral technology and investment-specific technology permanently affect TFP. Therefore, our UC framework provides a theoretical structure to derive sensible and equally valid long-run restrictions about RPI and TFP, in the spirit of Gali (1999) and Fisher (2006), to identify NT and IST shocks in a VAR framework.

[Figure 2 about here.]

[Table 1 about here.]

5 Model Comparison

In this section, we explore the fit of our bivariate unobserved components model defined in (1)–(6) to alternative restricted specifications that encompass the various assumptions in the theoretical and empirical literature. Therefore, our model constitutes a complete laboratory that may be used to evaluate the plausibility of competing specifications and their probable ramifications for business cycle analysis.

We adopt the Bayesian model comparison framework to compare various specifications via the Bayes factor. More specifically, suppose we wish to compare model M_0 against model M_1. Each model $M_i, i = 0, 1$, is formally defined by a likelihood function $p(\mathbf{y} \,|\, \boldsymbol{\theta}_i, M_i)$ and a prior distribution on the model-specific parameter vector $\boldsymbol{\theta}_i$ denoted by $p(\boldsymbol{\theta}_i \,|\, M_i)$. Then, the Bayes factor in favor of M_0 against M_1 is defined as

$$\mathrm{BF}_{01} = \frac{p(\mathbf{y} \,|\, M_0)}{p(\mathbf{y} \,|\, M_1)},$$

where $p(\mathbf{y} \,|\, M_i) = \int p(\mathbf{y} \,|\, \boldsymbol{\theta}_i, M_i) p(\boldsymbol{\theta}_i \,|\, M_i) \mathrm{d}\boldsymbol{\theta}_i$ is the marginal likelihood under model M_i, $i = 0, 1$. Note that the marginal likelihood is the marginal data density (unconditional on the prior distribution) implied by model M_i evaluated at the observed data $\mathbf{y}$. Since the marginal likelihood can be interpreted as a joint density forecast evaluated at the observed data, it has a built-in penalty for model complexity. If the observed data is likely under the model, the associated marginal likelihood would be "large" and vice versa. It follows that $\mathrm{BF}_{01} > 1$ indicates evidence in favor of model M_0 against M_1, and the weight of evidence is proportional to the value of the Bayes factor. For a textbook treatment of the Bayes factor and the computation of the marginal likelihood, see, Chan et al. (2019).

5.1 Testing $\gamma = 0$

In the first modified model, we impose the restriction that $\gamma = 0$, which essentially amounts to testing the GPT theory. If the restricted model were preferred over the unrestricted model, then the trends in RPI and TFP would be orthogonal, and the traditional approach of specifying neutral technology and investment-specific technology would be well founded.

The posterior mean of γ is estimated to be about 0.48, and the posterior standard deviation is 0.55. Most of the mass of the posterior distribution is on positive values—the posterior probability that $\gamma > 0$ is 0.88. To formally test if $\gamma = 0$, we compute the Bayes factor in favor of the baseline model in (1)–(6) against the unrestricted version with $\gamma = 0$ imposed. In this case the Bayes factor can be obtained by using the Savage-Dickey density ratio $p(\gamma = 0)/p(\gamma = 0 \,|\, \mathbf{y})$.[4]

The Bayes factor in favor of the baseline model is about 1.1, suggesting that even though there

[4]Earlier work such as Everaert (2011) has used unobserved components models to test for cointegration with the frequentist approach. In Appendix A.6 we provide Monte Carlo results to assess the empirical performance of this Bayesian testing procedure using the Bayes factor. Overall, the Monte Carlo experiments indicate that this approach works reasonably well.

is some evidence against the hypothesis $\gamma = 0$, the evidence is not strong. To better understand this result, we plot the prior and posterior distributions of γ in Figure 3. As is clear from the figure, the prior and posterior densities at $\gamma = 0$ have similar values. However, it is also apparent that the data moves the prior distribution to the right, making larger values of γ more likely under the posterior distribution. Hence, there seems to be some support for the hypothesis that $\gamma > 0$. More importantly, this result confirms the fact that $\gamma \neq 0$, and the approach of modelling neutral technology and investment-specific technology as following independent processes is clearly not supported by the data. As we amply discussed in Section 4.2, such common trend misspecification yield incorrect restrictions about balanced growth restrictions and invalid conclusions about the main sources of business cycle fluctuations. Therefore, this suggest that business cycle researchers need to modify the specification of DSGE models to consider the possibility of the existence of this long-run joint relationship observed in the data and its potential effect on fluctuations.

[Figure 3 about here.]

5.2 Testing $\varphi_\mu = \varphi_{\mu_x} = 0$

Next, we test the joint hypothesis that $\varphi_\mu = \varphi_{\mu_x} = 0$. In this case, the restricted model would allow the trend component to follow a random walk plus drift process such that the growths in RPI and TFP are constant. This would capture the assumptions in Fisher (2006).

Again the Bayes factor in favor of the baseline model against the restricted model with $\varphi_\mu = \varphi_{\mu_x} = 0$ imposed can be obtained by computing the Savage-Dickey density ratio $p(\varphi_\mu = 0, \varphi_{\mu_x} = 0)/p(\varphi_\mu = 0, \varphi_{\mu_x} = 0 \,|\, \mathbf{y})$. The Bayes factor in favor of the restricted model with $\varphi_\mu = \varphi_{\mu_x} = 0$ is about 14. This indicates that there is strong evidence in favor the hypothesis $\varphi_\mu = \varphi_{\mu_x} = 0$. This is consistent with the estimation results reported in Table 1—the estimates of φ_μ and φ_{μ_x} are both small in magnitude with relatively large posterior standard deviations. This is an example where the Bayes factor favors a simpler, more restrictive model.

Despite restricting $\varphi_\mu = \varphi_{\mu_x} = 0$, this restricted model is able to fit the RPI and TFP series very well, as shown in Figure 4. This suggests that an ARIMA(0,1,0) specification for neutral technology and investment-specific technology would also be a viable alternative in DSGE models. Nonetheless, an ARIMA(0,1,0) specification, which implies a constant growth rate, would be incapable of generating the positive autocorrelated AR(1) observed in output growth. As we mentioned in Section 4.1, Cogley and Nason (1995a) show that output dynamics are essentially determined by impulse dynamics. Therefore, DSGE models would be able to match the periodicity of output if the process of technology is simulated to have a positively autocorrelated growth rate. In addition, Cogley and Nason (1995a) demonstrate that an ARIMA(1,1,0) only would be needed to produce a humped-shaped response of output in order for models to match these facts about autocorrelated output growth.

[Figure 4 about here.]

5.3 Testing $\varphi_{\mu_x} = \sigma^2_{\eta_x} = 0$

Now, we test the joint hypothesis that $\varphi_{\mu_x} = \sigma^2_{\eta_x} = 0$. This restriction goes to the heart of the debate between Schmitt-Grohé and Uribè (2011) and Benati (2013). If the restricted model held true, then RPI and TFP would be co-integrated as argued by Schmitt-Grohé and Uribè (2011). On the other hand, if the restrictions were rejected, we would end with a situation where RPI and TFP are not co-integrated, but still share a common component as shown in Benati (2013).

Since zero is at the boundary of the parameter space of $\sigma^2_{\eta_x}$, the relevant Bayes factor cannot be computed using the Savage-Dickey density ratio.[5] Instead, we compute the log marginal likelihoods of the baseline model and the restricted version with $\varphi_{\mu_x} = \sigma^2_{\eta_x} = 0$. The marginal likelihoods of the two models are obtained by using the adaptive importance sampling estimator known as the cross-entropy method proposed in Chan and Eisenstat (2015).

The log marginal likelihood of the baseline model is 1684, compared to 1627 of the restricted version. This means that the log Bayes factor in favor of the baseline model is 57, suggesting overwhelming support for the unrestricted model. This shows that we can reject the joint hypothesis $\varphi_{\mu_x} = \sigma^2_{\eta_x} = 0$. This result along with the fact that $\gamma > 0$ implies a positive covariation between RPI and TFP. While this conclusion is identical to the result in Benati (2013), our approach sets itself apart in the sense that it provides a quantitative measure of the relationship between the

[5] An alternative approach to test the non-standard hypothesis $\sigma^2_{\eta_x} = 0$ is to use the non-centered parameterization of Frühwirth-Schnatter and Wagner (2010), i.e., reparameterzing the model in terms of σ_{η_x}. This approach has been adopted in, e.g., Berger, Everaert and Vierke (2016) and Chan (2018). In our setting, however, the state equations (3) and (4) have a second-order Markov structure with a deterministic trend component. As such, the non-centered parameterization approach, which is designed for random walk state equations, cannot be directly applied.

trends in RPI and TFP, and it offers an interpretation that may be easily traced to economic theory. Furthermore, it strengthens the argument suggested by many studies that DSGE models should be modified to incorporate the joint long-run relationship between neutral technology and investment-specific technology as misspecification may lead to misleading conclusions about sources of business cycle fluctuations. For instance, while both Schmitt-Grohé and Uribè (2011) and Benati (2013) adopt different strategies, the results in both studies appear to indicate (1) the presence of a common relationship between neutral technology and investment-specific technology (the authors differ in their views about the nature of this common relationship), and (2) the emergence of a new, common stochastic trend shocks, that play a more significant role than each technology shocks in the business cycle fluctuations of macro variables. Schmitt-Grohé and Uribè (2011) manage to incorporate this common relationship, in the form of a co-integration assumption, in a DSGE model to gauge the role of the new shocks, but Benati (2013) uses the co-variation result in a vectorautoregressive setting. Since our framework delivers results which seems to align with those in Benati (2013), we argue that our structural framework may help to guide the joint co-variation specification in DSGE models, in the spirit of Ferroni (2011) and Canova (2014).

5.4 Testing $\varphi_\mu = \varphi_{\mu_x} = 1$

Lastly, we test the joint hypothesis that $\varphi_\mu = \varphi_{\mu_x} = 1$. This restriction would imply that the trend components in RPI and TFP follow an ARIMA(0,2,0) process such that the growth rate of these trends is integrated of order one.

Since the value 1 is at the boundary of the parameter space of both φ_μ and φ_{μ_x}, the relevant Bayes factor cannot be computed using the Savage-Dickey density ratio. We instead compute the log marginal likelihoods of the baseline model and the restricted version with $\varphi_\mu = \varphi_{\mu_x} = 1$, again using the cross-entropy method in Chan and Eisenstat (2015). The former value is 1684 and the latter is 1034, suggesting overwhelming support for the baseline model. Hence, we can reject the joint hypothesis $\varphi_\mu = \varphi_{\mu_x} = 1$. These are consistent with the estimation results reported in Table 1. Specifically, under the baseline model, φ_μ and φ_{μ_x} are estimated to be, respectively, 0.101 and -0.051, and both values are far from 1. These results are quite reassuring because if the restricted model held true, then we may be tempted to impose an ARMA(0,2,0) specification for neutral technology and investment-specific technology. However, this will be hard to reconcile with the fact that output growth has been documented in many studies to follow a positively autocorrelated AR(1) process. Since output dynamics are essentially determined by impulse dynamics, an ARMA(0,2,0) process for technology would indicate that output growth is integrated of order one, violating that stylized fact.

6 Conclusion

The aim of the paper is to evaluate the relationship between the relative price of investment (RPI) and total factor productivity (TFP) to inform the specification of neutral technology and investment-specific technology in DSGE models. Using the unobserved component decomposition that allows us to separate a trend component from a cyclical component in a time series, we specify a model that features a common component between the trends component in RPI and TFP. The main results of the analysis is that RPI and TFP may each be generated by an ARIMA(1,1,0) process, and the two series share a common stochastic trend component that may drive the mean growth rate of aggregate productivity in the United States. As documented in many studies, we view this common stochastic trend component as capturing permanent changes in General Purpose Technology from innovations in information and communication technologies. In addition, our findings provide some guidance to the idiosyncratic and joint specification of neutral technology and investment-specific technology in DSGE models. This may help to reduce misspecification errors that contribute to misleading interpretations about the sources of business cycles and the analysis of output dynamics.

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Figure 1: Fitted values of $\widehat{z}_t = \tau_t$ and $\widehat{x}_t = \gamma\tau_t + \tau_{x,t}$. The shaded region represents the 5-th and 95-th percentiles.

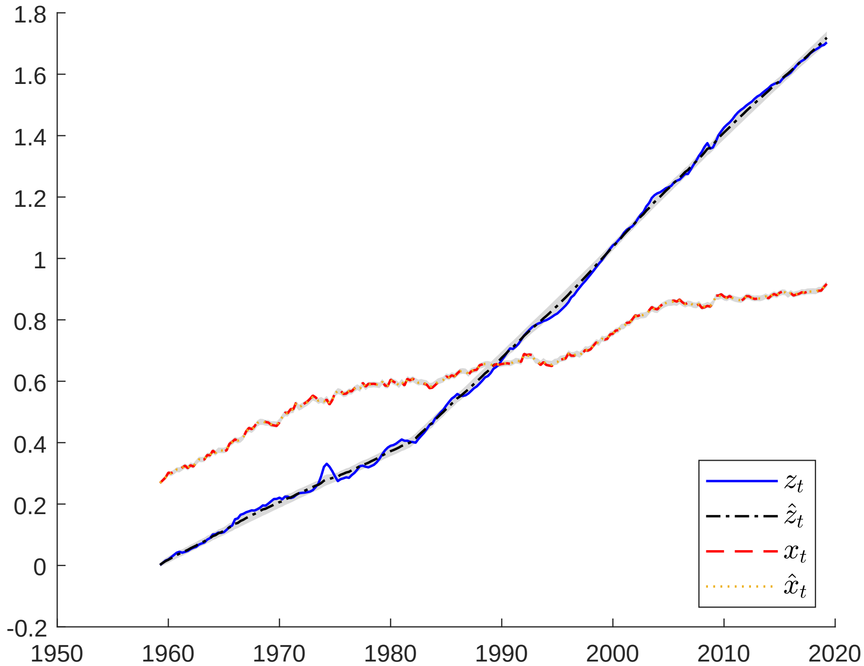

Figure 2: Posterior means of τ_t and $\tau_{x,t}$. The shaded region represents the 16-th and 84-th percentiles.

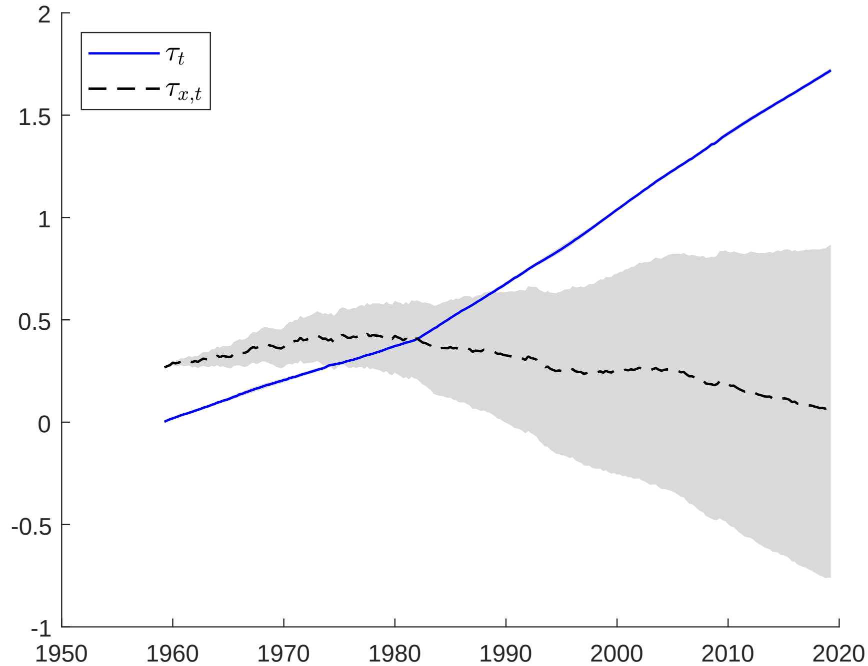

Figure 3: Prior and posterior distributions of γ.

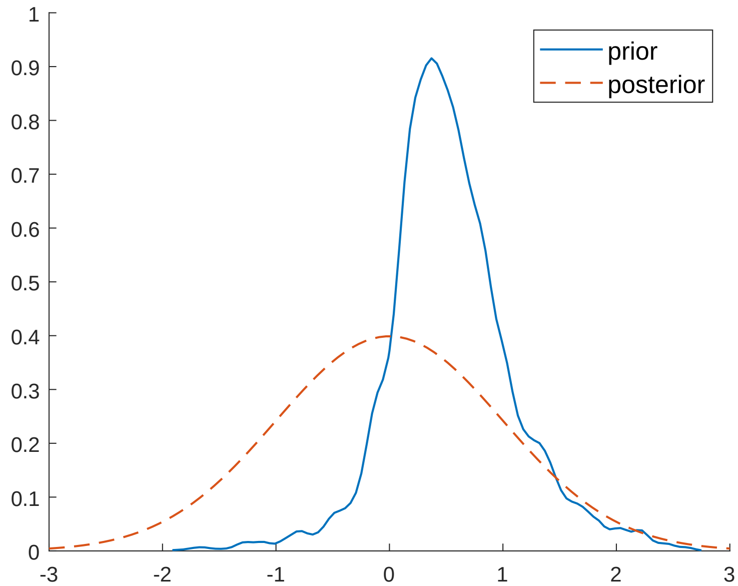

Figure 4: Fitted values of $\widehat{z}_t = \tau_t$ and $\widehat{x}_t = \gamma\tau_t + \tau_{x,t}$ of the restricted model with $\varphi_\mu = \varphi_{\mu_x} = 0$. The shaded region represents the 5-th and 95-th percentiles.

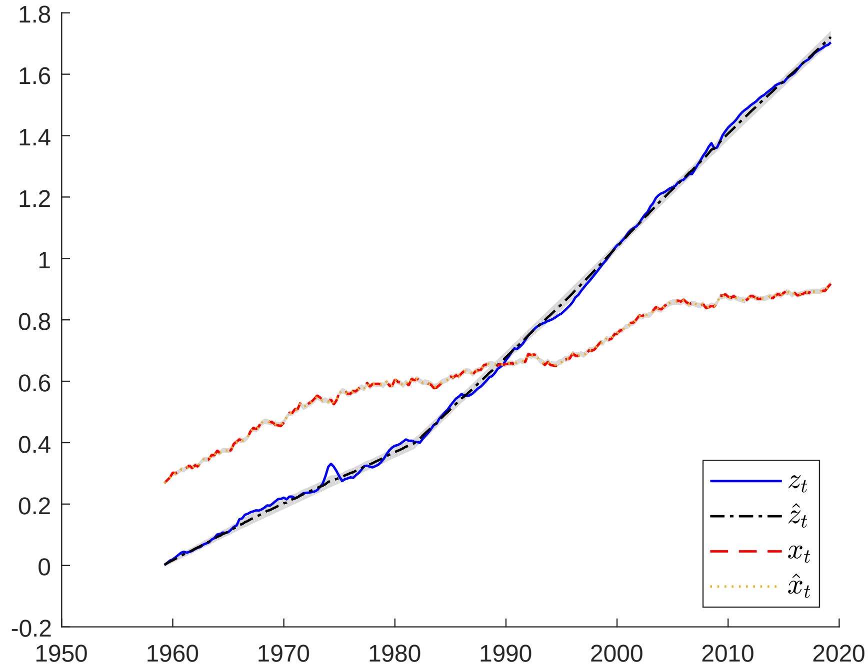

Table 1: Posterior means, standard deviations and 95% credible intervals of model parameters

	posterior mean	posterior std. dev.	95% credible interval
γ	0.478	0.549	$(-0.508, 1.797)$
ζ_1	0.004	0.0003	$(0.004, 0.005)$
ζ_2	0.009	0.0003	$(0.008, 0.009)$
$\zeta_{x,1}$	0.001	0.003	$(-0.004, 0.006)$
$\zeta_{x,2}$	-0.003	0.005	$(-0.014, 0.007)$
φ_μ	0.101	0.169	$(-0.087, 0.584)$
φ_{μ_x}	-0.051	0.072	$(-0.212, 0.076)$
$\phi_{z,1}$	1.458	0.074	$(1.308, 1.600)$
$\phi_{z,2}$	-0.568	0.084	$(-0.731, -0.408)$
$\phi_{x,1}$	0.913	0.342	$(0.178, 1.48)$
$\phi_{x,2}$	-0.240	0.263	$(-0.676, 0.314)$
σ_z^2	1.79×10^{-5}	4.31×10^{-6}	$(1.01 \times 10^{-5}, 2.61 \times 10^{-5})$
σ_x^2	7.76×10^{-6}	3.99×10^{-6}	$(3.09 \times 10^{-6}, 1.81 \times 10^{-5})$
σ_η^2	5.51×10^{-6}	3.43×10^{-6}	$(1.85 \times 10^{-7}, 1.26 \times 10^{-5})$
$\sigma_{\eta_x}^2$	5.17×10^{-5}	7.75×10^{-6}	$(3.56 \times 10^{-5}, 6.64 \times 10^{-5})$

A Appendix

A.1 Model economy

In this section, we use a neoclassical model similar to the structure in Greenwood et al. (1997), Oulton (2007), and Moura (2021) to derive our UC framework in equations (1)-(6). Consider the following model

$$
\begin{aligned}
Y_t &= C_t + I_t = z_t K_t^{\alpha} \big(X_t^z h_t \big)^{1-\alpha}, && \text{(A.1)} \\
I_t^* &= a_t X_t^a I_t && \text{(A.2)} \\
K_{t+1} &= (1-\delta) K_t + I_t^*. && \text{(A.3)}
\end{aligned}
$$

Equation (A.1) is the aggregate production function where output, (Y_t), which can be used for either consumption, (C_t), or gross investment, (I_t), is produced with capital, (K_t), and labor, (h_t) such that $0 \leq h_t \leq 1$. The production function is subjected to a stationary neutral technological shock, (a_t), and a non-stationary neutral technological shock, (X_t^z). Equation (A.2) relates investment in efficiency units, (I_t^*), to gross investment. The terms a_t and X_t^a denote, respectively, stationary and non-stationary investment-specific technology. Finally, equation (A.3) shows the evolution of the capital stock. It is important to note that output and gross investment are both measured in units of consumption goods.

In this model economy, Total Factor Productivity (TFP) and the Relative Price of Investment (RPI), denoted as P_t^I, are given, respectively, by

$$
\begin{aligned}
TFP_t &= \frac{Y_t}{K_t^{\alpha} h_t^{1-\alpha}} = z_t \big(X_t^z \big)^{1-\alpha} && \text{(A.4)} \\
P_t^I &= \frac{1}{a_t X_t^a}. && \text{(A.5)}
\end{aligned}
$$

Along a balanced growth path, the stochastic trend in Y_t and C_t is given by $X_t^z \big(X_t^a \big)^{\frac{\alpha}{1-\alpha}}$ while the stochastic trend in investment measured in efficiency units I_t^* is $X_t^z \big(X_t^a \big)^{\frac{1}{1-\alpha}}$.

Using the standard Divisia definition of aggregate output, output, Y_t^D, is well-approximated by a share weighted index[6]

$$Y_t^D = C_t^{1-\gamma} I_t^{*\gamma} \tag{A.6}$$

where γ is the current price share of investment in the value of output. Under this definition of output, TFP is defined as

$$TFP_t^D = \frac{Y_t^D}{K_t^\alpha h_t^{1-\alpha}} \tag{A.7}$$

Therefore, expressions (A.4) and (A.7) may be combined to yield

$$TFP_t^D = z_t \big(X_t^z\big)^{1-\alpha} \frac{Y_t^D}{Y_t}, \tag{A.8}$$

To determine the stochastic balanced growth path in TFP_t^D, we start with expression (A.8) and apply the logarithm on both sides,

$$\log(TFP_t^D) = \log(z_t) + (1-\alpha)\log(X_t^z) + \log(Y_t^D) - \log(Y_t),$$

then we take the first difference, and use the definition in A.6 to get

$$\begin{aligned}
\log(TFP_t^D) - \log(TFP_{t-1}^D) &= (1-\alpha)\Big(\log X_t^z - \log X_{t-1}^z\Big) + (1-\gamma)\Big(\log C_t - \log C_{t-1}\Big) \\
&\quad +\gamma\Big(\log I_t^* - \log I_{t-1}^*\Big) - \Big(\log Y_t - \log Y_{t-1}\Big) \\
&= (1-\alpha)\Big(\log X_t^z - \log X_{t-1}^z\Big) \\
&\quad +(1-\gamma)\Big[\log\Big(X_t^z (X_t^a)^{\frac{\alpha}{1-\alpha}}\Big) - \log\Big(X_{t-1}^z (X_{t-1}^a)^{\frac{\alpha}{1-\alpha}}\Big)\Big] \\
&\quad +\gamma\Big[\log\Big(X_t^z (X_t^a)^{\frac{1}{1-\alpha}}\Big) - \log\Big(X_{t-1}^z (X_{t-1}^a)^{\frac{1}{1-\alpha}}\Big)\Big] \\
&\quad -\Big[\log\Big(X_t^z (X_t^a)^{\frac{\alpha}{1-\alpha}}\Big) - \log\Big(X_{t-1}^z (X_{t-1}^a)^{\frac{\alpha}{1-\alpha}}\Big)\Big] \\
&= (1-\alpha)\Big(\log X_t^z - \log X_{t-1}^z\Big) + \gamma\Big[\log X_t^a - \log X_{t-1}^a\Big]
\end{aligned}$$

where we use the fact that, along a balanced growth path, Y_t and C_t grow at the same rate as

[6] The Divisia index for output is $\Delta \log Y_t^D = (1-\gamma)\Delta \log C_t + \gamma \Delta \log I_t^*$. Without loss of generality, we may normalize the levels of $\log Y_t^D$, $\log C_t$, and $\log I_t^*$ at period 0 to be zero, and rearrange to obtain the expression for Y_t^D. We use the superscript "D" to emphasize the dependence of output on the Divisia definition. Also, Y_t^D differs from Y_t in the sense that the latter is measured in consumption units.

$X_t^z\left(X_t^a\right)^{\frac{\alpha}{1-\alpha}}$ while I_t^* grows at the faster rate $X_t^z\left(X_t^a\right)^{\frac{1}{1-\alpha}}$. Then, using a simple change of variable to specify technology so that it is Hicks neutral instead of Harrod neutral, i.e. $\widetilde{X}_t^z = \left(X_t^z\right)^{1-\alpha}$, we obtain

$$\begin{aligned}
\log(TFP_t^D) - \log(TFP_{t-1}^D) &= \left(\log \widetilde{X}_t^z - \log \widetilde{X}_{t-1}^z\right) + \gamma\left[\log X_t^a - \log X_{t-1}^a\right] && \text{(A.9)} \\
\log(TFP_t^D) - \log \widetilde{X}_t^z - \gamma \log X_t^a &= \log(TFP_{t-1}^D) - \log \widetilde{X}_{t-1}^z - \gamma \log X_{t-1}^a && \text{(A.10)} \\
\frac{TFP_t^D}{\widetilde{X}_t^z\left(X_t^a\right)^\gamma} &= \frac{TFP_{t-1}^D}{\widetilde{X}_{t-1}^z\left(X_{t-1}^a\right)^\gamma}, && \text{(A.11)}
\end{aligned}$$

or in other words, along the balanced growth path, $\frac{TFP_t^D}{\widetilde{X}_t^z\left(X_t^a\right)^\gamma}$ is stationary. We may apply the same logic to the relative price of investment, P_t^I, to show that $P_t^I X_t^a$ is also stationary along the stochastic balanced growth path. Let tfp_t^D and p_t^I denote stationary TFP and stationary RPI, respectively; therefore, applying the logarithm and rearranging yields the system

$$\begin{aligned}
\log P_t^I &= \log X_t^a + \log p_t^I && \text{(A.12)} \\
\log TFP_t^D &= \gamma \log X_t^a + \log \widetilde{X}_t^z + \log tfp_t^D && \text{(A.13)}
\end{aligned}$$

Suppose that we further assume that the growth rate of the non-stationary neutral technology, and the non-stationary investment-specific technology, denoted as $\mu_t^z \equiv \Delta \log \widetilde{X}_t^z$ and $\mu_t^a \equiv \Delta \log X_t^a$, respectively, follow stationary $AR(1)$ processes, and the stationary components of TFP and RPI, $\log tfp_t^D$ and $\log p_t^I$, follow $AR(1)$ processes as we specify in the paper. Therefore, equations (A.12), (A.13) along with the assumed specifications constitute the full characterization of our UC framework illustrated in equations (1)-(6) within the manuscript.

We may also use our framework to derive expressions for RPI and TFP in terms of neutral technology and investment-specific technology shocks to obtain sensible restrictions for the identification of these shocks in a Vector Autoregressive (VAR) framework. We will assume without loss of generality that both neutral technology and investment-specific technology follow a unit root process such that $\widetilde{X}_t^z = \widetilde{X}_{t-1}^z \exp(\varepsilon_t^z)$ and $X_t^a = X_{t-1}^a \exp(\varepsilon_t^a)$. In that case, ε_t^z and ε_t^a are the neutral technology shocks and the investment-specific technology shocks, respectively.

For RPI, we apply the log to expression A.5 and take the first difference to get:

$$\begin{aligned}
\log P_t^I - \log P_{t-1}^I &= -\log X_t^a + \log X_{t-1}^a - \log a_t + \log a_{t-1} \\
\log P_t^I - \log P_{t-1}^I &= -\log X_t^a + \log X_{t-1}^a \\
\Delta \log P_t^I &= -\varepsilon_t^a
\end{aligned}$$

Note that since a_t follows stationary process, its growth rate, $\log a_t - \log a_{t-1}$, is zero.

We apply the same logic to TFP in the case when output is expressed in consumption units. Specifically, applying the log to expression A.4 and taking the first difference yields

$$\begin{aligned}
\log TFP_t - \log TFP_{t-1} &= \log \widetilde{X}_t^z - \log \widetilde{X}_{t-1}^z + \log z_t - \log z_{t-1} \\
\log TFP_t - \log TFP_{t-1} &= \log \widetilde{X}_t^z - \log \widetilde{X}_{t-1}^z \\
\Delta \log TFP_t &= \varepsilon_t^z
\end{aligned}$$

where we assume again that the growth rate of z_t is zero. Note that technology is Hicks neutral, i.e. $\widetilde{X}_t^z = \left(X_t^z\right)^{1-\alpha}$.

In the case when output is measured via the Divisia index, we rely on expression A.9 to obtain

$$\begin{aligned}
\log(TFP_t^D) - \log(TFP_{t-1}^D) &= \left(\log \widetilde{X}_t^z - \log \widetilde{X}_{t-1}^z\right) + \gamma\left[\log X_t^a - \log X_{t-1}^a\right] \\
\Delta \log(TFP_t^D) &= \varepsilon_t^z + \gamma \varepsilon_t^a
\end{aligned}$$

In sum, the expressions for RPI and TFP in terms of the neutral technology and investment-specific technology shocks are given by

$$\Delta \log(PI_t) = -\varepsilon_t^a \tag{A.14}$$

$$\Delta \log(TFP_t) = \varepsilon_t^z, \text{with output in consumption units} \tag{A.15}$$

$$\Delta \log(TFP_t^D) = \varepsilon_t^z + \gamma \varepsilon_t^a, \text{with output via Divisia index.} \tag{A.16}$$

From expression A.14, it is evident that a positive IST shock ($\varepsilon_t^a > 0$) has a permanent impact on the log level of the relative price of investment. When output is measured in consumption units, expression (A.15) shows that only neutral technology affects the log level TFP permanently. With Divisia-index output, both neutral technology and investment-specific technology affect TFP. Therefore, the definition of output matters for the identification of neutral technology shocks and investment-specific technology in a bivariate system consisting of the log difference of RPI and the log difference of TFP.

A.2 Common Trend in RPI and TFP

In this section, we empirically revisit two issues that have been explored in both Schmitt-Grohé and Uribè (2011) and Benati (2013). The first is to determine whether RPI and TFP possess each a stochastic non-stationary component, and the second is to assess whether these non-stationary components are related. Therefore, we perform unit root tests and co-integration tests using U.S. data over the period $1959 : Q2$ to $2019.Q2$ using the logarithm of RPI and the logarithm of TFP.

A.3 Unit root tests

We carry out Augmented Dickey Fuller (ADF) tests that examine the null hypothesis that the logarithms of RPI and TFP have a unit root. The lag order is chosen based on the Schwartz and Hannah-Quinn criteria (SIC and HQ, respectively).[7] The results of the tests are presented in Table 2, and they clearly indicate that the tests fail to reject the null hypothesis of the presence of a unit root in the series at the standard 5% confidence level.

An alternative to the ADF test is the Kwiatkowski, Phillips, Schmidt, and Shin (KPSS) test that evaluates the null hypothesis that the time series is stationary in levels. The lag length is still selected according to the SIC and HQ criteria, and we allow for the possibility of a time trend in the series. The results are illustrated in Table 3, and they are consistent with those obtained from ADF tests: the KPSS rejects the null hypothesis of stationarity in the logarithm of the relative price of investment and total factor productivity.

Overall, stationarity tests are unequivocal in terms of the univariate properties of RPI and TFP: both time series contain a stochastic non-stationary component.

[Table 2 about here.]

[7] We use JMulti to determine the optimal lag order for each variable. We set 10 as the maximum number of endogenous variables. The results indicate that 1 or 2 might be optimal. We obtained the same results using codes provided by Benati (2013).

[Table 3 about here.]

A.4 Co-integration between RPI and TFP

The results from the previous section clearly indicate that the time series of both RPI and TFP contain a non-stationary stochastic component. With that information at hand, we assess the extent to which these two non-stochastic components might be co-integrated. Specifically, we perform the Johansen's trace test that evaluates the null hypothesis that there is no co-integration relationship between the two series. A rejection of this hypothesis would indicate that RPI and TFP are driven by a single stochastic component. Consistent with the previous section, we still select the lag length according to the SIC and HQ criteria, and the result points to 1 as the optimal lag order. As discussed in Benati (2013), the lag order selection for VARs containing integrated variables in the Johansen's procedure may greatly affect the results of the test. Therefore, we also consider lag orders of 7 and 3 as alternatives, as in Schmitt-Grohé and Uribè (2011) and Benati (2013), respectively. A final and crucial component of the test is the specification of the data generating process (DGP) for the co-integrated model as this step has great importance on the results of the test. In other words, should a deterministic term, constant or linear term, be included in the DGP? Such decisions are usually guided by the underlying process of the variables which may or may not contain a drift term. Both the inverse of RPI and TFP appear to be trending upward, hence we consider the addition of a constant and linear term in the DGP. The results of the Johansen's trace tests are shown in Table 4.

[Table 4 about here.]

The co-integration results are inconclusive. When the lag order is 7, the null hypothesis of a zero co-integrating vector is rejected at the standard 5% confidence level, a result that echoes those in Benati (2013) and Schmitt-Grohé and Uribè (2011). However, the test fails to reject the null hypothesis of a zero co-integrating vector when a lag order of 1 or 3 is considered. Therefore, it appears that it is impossible to claim with certainty that RPI and TFP are driven by a single stochastic trend component.

A.5 Estimation Details

In this appendix we provide the estimation details of the bivariate unobserved components model specified in the main text. For convenience we reproduce the UC model below:

$$z_t = \tau_t + c_{z,t}, \tag{A.1}$$

$$x_t = \gamma\tau_t + \tau_{x,t} + c_{x,t}, \tag{A.2}$$

where τ_t and $\tau_{x,t}$ are the trend components, whereas $c_{z,t}$ and $c_{x,t}$ are the transitory components. The transitory components are assumed to follow the AR(2) processes below:

$$c_{z,t} = \phi_{z,1}c_{z,t-1} + \phi_{z,2}c_{z,t-2} + \epsilon_{z,t}, \tag{A.3}$$

$$c_{x,t} = \phi_{x,1}c_{x,t-1} + \phi_{x,2}c_{x,t-2} + \epsilon_{x,t}, \tag{A.4}$$

where $\epsilon_{z,t} \sim \mathcal{N}(0, \sigma_z^2)$ and $\epsilon_{x,t} \sim \mathcal{N}(0, \sigma_x^2)$, and the initial conditions $c_{z,0}, c_{x,0}, c_{z,-1}$ and $c_{x,-1}$ are assumed to be zero. For the trend components, we model the first differences of $\Delta\tau_t$ and $\Delta\tau_{x,t}$ as stationary processes, each with a break at $t = T_B$ with a different unconditional mean. More specifically, consider:

$$\Delta\tau_t = (1-\varphi_\mu)\zeta_1 1(t < T_B) + (1-\varphi_\mu)\zeta_2 1(t \geq T_B) + \varphi_\mu\Delta\tau_{t-1} + \eta_t, \tag{A.5}$$

$$\Delta\tau_{x,t} = (1-\varphi_{\mu_x})\zeta_{x,1} 1(t < T_B) + (1-\varphi_{\mu_x})\zeta_{x,2} 1(t \geq T_B) + \varphi_{\mu_x}\Delta\tau_{x,t-1} + \eta_{x,t}, \tag{A.6}$$

where $1(\cdot)$ denotes the indicator function, $\eta_t \sim \mathcal{N}(0, \sigma_\eta^2)$ and $\eta_{x,t} \sim \mathcal{N}(0, \sigma_{\eta_x}^2)$ are independent of each other at all leads and lags. The initial conditions $\boldsymbol{\tau}_0 = (\tau_0, \tau_{-1}, \tau_{x,0}, \tau_{x,-1})'$ are treated as unknown parameters.

Section 3 in the main text outlines a 5-block posterior simulator to estimate the above bivariate unobserved components model. Below we describe the implementation details of all the steps.

Step 1. Since $\boldsymbol{\tau}$ and γ enter the likelihood multiplicatively, we sample them jointly to improve efficiency of the posterior sampler. In particular, we first sample γ marginally of $\boldsymbol{\tau}$, followed by drawing $\boldsymbol{\tau}$ conditional on the sampled γ. The latter step is straightforward as the model specified in (1)–(6) defines a linear state space model for $\boldsymbol{\tau}$. In what follows we derive the full conditional distribution of $p(\boldsymbol{\tau} \,|\, \mathbf{y}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0)$. Then, we outline a Metropolis-Hastings algorithm to sample γ marginally of $\boldsymbol{\tau}$.

To derive the conditional distribution $p(\boldsymbol{\tau} \,|\, \mathbf{y}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0)$, note that by Bayes' theorem we have

$$p(\boldsymbol{\tau} \,|\, \mathbf{y}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0) \propto p(\mathbf{y} \,|\, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\sigma}^2) p(\boldsymbol{\tau} \,|\, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0),$$

where the conditional likelihood $p(\mathbf{y} \,|\, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\sigma}^2)$ and the prior $p(\boldsymbol{\tau} \,|\, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0)$ are, respectively, defined by the observation equations (1), (2), (5) and (6) and the state equations (3)-(4).

First, we derive the conditional likelihood $p(\mathbf{y} \,|\, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\sigma}^2)$. Letting $\mathbf{c} = (c_{z,1}, c_{x,1}, \ldots, c_{z,T}, c_{x,T})$' we can stack the observation equations (1)–(2) over $t = 1, \ldots, T$ to get

$$\mathbf{y} = \boldsymbol{\Pi}_\gamma \boldsymbol{\tau} + \mathbf{c},$$

where $\boldsymbol{\Pi}_\gamma = \mathbf{I}_T \otimes \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$. Here $\mathbf{I}_T$ is the T-dimensional identity matrix and $\otimes$ is the Kronecker product. Next, we stack (5)–(6) over $t = 1, \ldots, T$ to get

$$\mathbf{H}_{\boldsymbol{\phi}} \mathbf{c} = \boldsymbol{\epsilon},$$

where $\boldsymbol{\epsilon} = (\epsilon_{z,1}, \epsilon_{x,1}, \ldots, \epsilon_{z,T}, \epsilon_{x,T})'$ and

$$\mathbf{H}_{\boldsymbol{\phi}} = \begin{pmatrix} \mathbf{I}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{A}_1 & \mathbf{I}_2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{I}_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{I}_2 & \cdots & \mathbf{0} \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{I}_2 \end{pmatrix}$$

with $\mathbf{A}_1 = \begin{pmatrix} -\phi_{z,1} & 0 \\ 0 & -\phi_{x,1} \end{pmatrix}$ and $\mathbf{A}_2 = \begin{pmatrix} -\phi_{z,2} & 0 \\ 0 & -\phi_{x,2} \end{pmatrix}$. Since the determinant of $\mathbf{H}_{\boldsymbol{\phi}}$ is one for any $\boldsymbol{\phi}$, it is invertible. It follows that $(\mathbf{c} \,|\, \boldsymbol{\phi}, \boldsymbol{\sigma}^2) \sim \mathcal{N}(\mathbf{0}, (\mathbf{H}_{\boldsymbol{\phi}}' \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1} \mathbf{H}_{\boldsymbol{\phi}})^{-1})$, where $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} = \text{diag}(\sigma_z^2, \sigma_x^2, \ldots, \sigma_z^2, \sigma_x^2)$.

It then follows that

$$(\mathbf{y}\,|\,\boldsymbol{\tau},\gamma,\boldsymbol{\phi},\boldsymbol{\sigma}^2)\sim\mathcal{N}(\mathbf{\Pi}_\gamma\boldsymbol{\tau},(\mathbf{H}_{\boldsymbol{\phi}}'\mathbf{\Sigma}_{\boldsymbol{\epsilon}}^{-1}\mathbf{H}_{\boldsymbol{\phi}})^{-1}). \tag{A.7}$$

Next, we derive the prior $p(\boldsymbol{\tau}\,|\,\boldsymbol{\varphi},\boldsymbol{\zeta},\boldsymbol{\sigma}^2,\boldsymbol{\tau}_0)$. To that end, construct the $T\times 1$ vector of indicators $\mathbf{d}_0=(1(1<T_B),1(2<T_B),\ldots,1(T<T_B))'$, and similarly define $\mathbf{d}_1$. Moreover, let

$$\widetilde{\boldsymbol{\mu}}_{\boldsymbol{\tau}}=\mathbf{d}_0\otimes\begin{pmatrix}(1-\varphi_\mu)\zeta_1\\(1-\varphi_{\mu_x})\zeta_{x,1}\end{pmatrix}+\mathbf{d}_1\otimes\begin{pmatrix}(1-\varphi_\mu)\zeta_2\\(1-\varphi_{\mu_x})\zeta_{x,2}\end{pmatrix}+\begin{pmatrix}(1+\varphi_\mu)\tau_0-\varphi_\mu\tau_{-1}\\(1+\varphi_{\mu_x})\tau_{x,0}-\varphi_{\mu_x}\tau_{x,-1}\\-\varphi_\mu\tau_0\\-\varphi_{\mu_x}\tau_{x,0}\\0\\\vdots\\0\end{pmatrix}.$$

Then, we can stack the state equations (A.5)–(A.6) over $t=1,\ldots,T$ to get

$$\mathbf{H}_{\boldsymbol{\varphi}}\boldsymbol{\tau}=\widetilde{\boldsymbol{\mu}}_{\boldsymbol{\tau}}+\boldsymbol{\eta},$$

where $\boldsymbol{\eta}=(\eta_1,\eta_{x,1},\ldots,\eta_T,\eta_{x,T})'$ and

$$\mathbf{H}_{\boldsymbol{\varphi}}=\begin{pmatrix}\mathbf{I}_2&\mathbf{0}&\mathbf{0}&\mathbf{0}&\cdots&\mathbf{0}\\\mathbf{B}_1&\mathbf{I}_2&\mathbf{0}&\mathbf{0}&\cdots&\mathbf{0}\\\mathbf{B}_2&\mathbf{B}_1&\mathbf{I}_2&\mathbf{0}&\cdots&\mathbf{0}\\\mathbf{0}&\mathbf{B}_2&\mathbf{B}_1&\mathbf{I}_2&\cdots&\mathbf{0}\\\vdots&&\ddots&\ddots&\ddots&\vdots\\\mathbf{0}&\cdots&\mathbf{0}&\mathbf{B}_2&\mathbf{B}_1&\mathbf{I}_2\end{pmatrix}$$

with $\mathbf{B}_1=\begin{pmatrix}-(1+\varphi_\mu)&0\\0&-(1+\varphi_{\mu_x})\end{pmatrix}$ and $\mathbf{B}_2=\begin{pmatrix}\varphi_\mu&0\\0&\varphi_{\mu_x}\end{pmatrix}$. Since the determinant of $\mathbf{H}_{\boldsymbol{\varphi}}$ is one for any $\boldsymbol{\varphi}$, it is invertible. It then follows that

$$(\boldsymbol{\tau}\,|\,\boldsymbol{\varphi},\boldsymbol{\zeta},\boldsymbol{\sigma}^2,\boldsymbol{\tau}_0)\sim\mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{\tau}},(\mathbf{H}_{\boldsymbol{\varphi}}'\mathbf{\Sigma}_{\boldsymbol{\eta}}^{-1}\mathbf{H}_{\boldsymbol{\varphi}})^{-1}), \tag{A.8}$$

where $\boldsymbol{\mu_\tau} = \mathbf{H}_{\boldsymbol{\varphi}}^{-1}\widetilde{\boldsymbol{\mu}}_{\boldsymbol{\tau}}$ and $\boldsymbol{\Sigma}_\eta = \mathrm{diag}(\sigma_\eta^2, \sigma_{\eta x}^2, \ldots, \sigma_\eta^2, \sigma_{\eta x}^2)$. Combining (A.7) and (A.8) and using standard regression results (see, e.g., Chan et al. 2019, pp.217-219), we have

$$(\boldsymbol{\tau} \,|\, \mathbf{y}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0) \sim \mathcal{N}(\widehat{\boldsymbol{\tau}}, \mathbf{K}_{\boldsymbol{\tau}}^{-1}),$$

where

$$\mathbf{K}_{\boldsymbol{\tau}} = \mathbf{H}_{\boldsymbol{\varphi}}'\boldsymbol{\Sigma}_{\boldsymbol{\eta}}^{-1}\mathbf{H}_{\boldsymbol{\varphi}} + \boldsymbol{\Pi}_\gamma'\mathbf{H}_{\boldsymbol{\phi}}'\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}\mathbf{H}_{\boldsymbol{\phi}}\boldsymbol{\Pi}_\gamma, \quad \widehat{\boldsymbol{\tau}} = \mathbf{K}_{\boldsymbol{\tau}}^{-1}\left(\mathbf{H}_{\boldsymbol{\varphi}}'\boldsymbol{\Sigma}_{\boldsymbol{\eta}}^{-1}\mathbf{H}_{\boldsymbol{\varphi}}\boldsymbol{\mu}_{\boldsymbol{\tau}} + \boldsymbol{\Pi}_\gamma'\mathbf{H}_{\boldsymbol{\phi}}'\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}\mathbf{H}_{\boldsymbol{\phi}}\mathbf{y}\right).$$

Since the precision matrix $\mathbf{K}_{\boldsymbol{\tau}}$ is banded—i.e., it is sparse and its nonzero elements are arranged along the main diagonal—one can sample from $p(\boldsymbol{\tau} \,|\, \mathbf{y}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0)$ efficiently using the precision sampler in Chan and Jeliazkov (2009).

Next, we outline a Metropolis-Hastings algorithm to sample γ marginally of $\boldsymbol{\tau}$. For that we need to evaluate the integrated likelihood

$$p(\mathbf{y} \,|\, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0) = \int_{\mathbb{R}^{2T}} p(\mathbf{y} \,|\, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\sigma}^2) p(\boldsymbol{\tau} \,|\, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0)\mathrm{d}\boldsymbol{\tau}.$$

Traditionally, this is done by using the Kalman filter. However, it turns out that we can obtain an analytical expression of the integrated likelihood and evaluate it efficiently using band matrix routines. Using a similar derivation in Chan and Grant (2016), one can show that

$$\begin{aligned} &p(\mathbf{y} \,|\, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0) \\ &\quad = (2\pi)^{-T}|\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}|^{-\frac{1}{2}}|\boldsymbol{\Sigma}_{\boldsymbol{\eta}}|^{-\frac{1}{2}}|\mathbf{K}_{\boldsymbol{\tau}}|^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}\left(\mathbf{y}'\mathbf{H}_{\boldsymbol{\phi}}'\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}\mathbf{H}_{\boldsymbol{\phi}}\mathbf{y} + \boldsymbol{\mu}_{\boldsymbol{\tau}}'\mathbf{H}_{\boldsymbol{\varphi}}'\boldsymbol{\Sigma}_{\boldsymbol{\eta}}^{-1}\mathbf{H}_{\boldsymbol{\varphi}}\boldsymbol{\mu}_{\boldsymbol{\tau}} - \widehat{\boldsymbol{\tau}}'\mathbf{K}_{\boldsymbol{\tau}}\widehat{\boldsymbol{\tau}}\right)}. \end{aligned} \tag{A.9}$$

The above expression involves a few large matrices, but they are all banded. Consequently, it can be evaluated efficiently using band matrix algorithms; see Chan and Grant (2016) for technical details.

Finally, we implement a Metropolis-Hastings step to sample γ with the Gaussian proposal $\mathcal{N}(\widehat{\gamma}, K_\gamma^{-1})$, where $\widehat{\gamma}$ is the mode of $\log p(\mathbf{y} \,|\, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0)$ and K_γ is the negative Hessian evaluated at the mode.

Step 2. To sample $\boldsymbol{\phi}$, we write (A.3)–(A.4) as a regression with coefficient vector $\boldsymbol{\phi}$:

$$\mathbf{c} = \mathbf{X}_{\boldsymbol{\phi}}\boldsymbol{\phi} + \boldsymbol{\epsilon},$$

where $\mathbf{c} = (c_{z,1}, c_{x,1}, \ldots, c_{z,T}, c_{x,T})'$ and $\mathbf{X}_{\boldsymbol{\phi}}$ is a $2T{\times}4$ matrix consisting of lagged values of $(c_{z,t}, c_{x,t})$. Then, by standard regression results, we have

$$(\boldsymbol{\phi} \,|\, \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0) \sim \mathcal{N}(\widehat{\boldsymbol{\phi}}, \mathbf{K}_{\boldsymbol{\phi}}^{-1}) 1(\boldsymbol{\phi} \in \mathbf{R}),$$

where

$$\mathbf{K}_{\boldsymbol{\phi}} = \mathbf{V}_{\boldsymbol{\phi}}^{-1} + \mathbf{X}_{\boldsymbol{\phi}}' \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1} \mathbf{X}_{\boldsymbol{\phi}}, \quad \widehat{\boldsymbol{\phi}} = \mathbf{K}_{\boldsymbol{\phi}}^{-1} \left(\mathbf{V}_{\boldsymbol{\phi}}^{-1} \boldsymbol{\phi}_0 + \mathbf{X}_{\boldsymbol{\phi}}' \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1} \mathbf{c} \right).$$

A draw from this truncated normal distribution can be obtained by the acceptance-rejection method, i.e., keep sampling from $\mathcal{N}(\widehat{\boldsymbol{\phi}}, \mathbf{K}_{\boldsymbol{\phi}}^{-1})$ until $\boldsymbol{\phi} \in \mathbf{R}$.

Step 3. Next, we simulate from $p(\boldsymbol{\varphi} \,|\, \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0)$. As in Step 2, we first write (A.5)–(A.6) as a regression with coefficient vector $\boldsymbol{\varphi}$:

$$\Delta \boldsymbol{\tau} = \boldsymbol{\mu}_{\boldsymbol{\varphi}} + \mathbf{X}_{\boldsymbol{\varphi}} \boldsymbol{\varphi} + \boldsymbol{\eta},$$

where $\Delta\boldsymbol{\tau} = (\Delta\tau_1, \Delta\tau_{x,1}, \ldots, \Delta\tau_T, \Delta\tau_{x,T})'$, $\boldsymbol{\mu}_{\boldsymbol{\varphi}} = (\zeta_1 1(1 < T_B) + \zeta_2 1(1 \geq T_B), \ldots, \zeta_1 1(T < T_B) + \zeta_2 1(T \geq T_B))'$ and

$$\mathbf{X}_{\boldsymbol{\varphi}} = \begin{pmatrix} \Delta\tau_0 - \zeta_1 1(1 < T_B) - \zeta_2 1(1 \geq T_B) & 0 \\ 0 & \Delta\tau_{x,0} - \zeta_{x,1} 1(1 < T_B) - \zeta_{x,2} 1(1 \geq T_B) \\ \vdots & \vdots \\ \Delta\tau_{T-1} - \zeta_1 1(T < T_B) - \zeta_2 1(T \geq T_B) & 0 \\ 0 & \Delta\tau_{x,T-1} - \zeta_{x,1} 1(T < T_B) - \zeta_{x,2} 1(T \geq T_B) \end{pmatrix}.$$

Again, by standard regression results, we have

$$(\boldsymbol{\varphi} \,|\, \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0) \sim \mathcal{N}(\widehat{\boldsymbol{\varphi}}, \mathbf{K}_{\boldsymbol{\varphi}}^{-1}) 1(\boldsymbol{\varphi} \in \mathbf{R}),$$

where

$$\mathbf{K}_{\boldsymbol{\varphi}} = \mathbf{V}_{\boldsymbol{\varphi}}^{-1} + \mathbf{X}_{\boldsymbol{\varphi}}' \boldsymbol{\Sigma}_{\boldsymbol{\eta}}^{-1} \mathbf{X}_{\boldsymbol{\varphi}}, \quad \widehat{\boldsymbol{\varphi}} = \mathbf{K}_{\boldsymbol{\varphi}}^{-1} \left(\mathbf{V}_{\boldsymbol{\varphi}}^{-1} \boldsymbol{\varphi}_0 + \mathbf{X}_{\boldsymbol{\varphi}}' \boldsymbol{\Sigma}_{\boldsymbol{\eta}}^{-1} (\Delta\boldsymbol{\tau} - \boldsymbol{\mu}_{\boldsymbol{\varphi}}) \right).$$

A draw from this truncated normal distribution can be obtained by the acceptance-rejection method.

Step 4. To implement Step 4, note that $\sigma_\eta^2, \sigma_{\eta_x}^2, \sigma_z^2, \sigma_x^2$ are conditionally independent given $\boldsymbol{\tau}$

and other parameters. Moreover, since the priors on σ_z^2 and σ_x^2 are inverse-gamma, so are the full posterior conditional distributions:

$$(\sigma_z^2 \,|\, \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\tau}_0) \sim \mathcal{IG}\left(\nu_z + \frac{T}{2}, S_z + \frac{1}{2}\sum_{t=1}^{T} \epsilon_{z,t}^2\right),$$
$$(\sigma_x^2 \,|\, \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\tau}_0) \sim \mathcal{IG}\left(\nu_x + \frac{T}{2}, S_x + \frac{1}{2}\sum_{t=1}^{T} \epsilon_{x,t}^2\right).$$

For σ_η^2 and $\sigma_{\eta_x}^2$, recall that they have gamma priors: $\sigma_\eta^2 \sim \mathcal{G}(1/2, 1/(2V_{\sigma_\eta}))$ and $\sigma_{\eta_x}^2 \sim \mathcal{G}(1/2, 1/(2V_{\sigma_{\eta_x}}))$. Hence, the full conditional density of σ_η^2 is given by

$$p(\sigma_\eta^2 \,|\, \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\tau}_0) \propto (\sigma_\eta^2)^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2V_{\sigma_\eta}}\sigma_\eta^2} \times (\sigma_\eta^2)^{-\frac{T}{2}} \mathrm{e}^{-\frac{1}{2\sigma_\eta^2}\sum_{t=1}^{T}\eta_t^2},$$

which is not a standard distribution. However, we can sample σ_η^2 via a Metropolis-Hastings step. Specifically, we first obtain a candidate draw σ_η^{2*} from the proposal distribution $\sigma_\eta^{2*} \sim \mathcal{IG}((T-1)/2, \sum_{t=1}^{T}\eta_t^2/2)$. Then, given the current draw σ_η^2, we accept the candidate with probability

$$\min\left\{1, \frac{\mathrm{e}^{\frac{1}{2V_{\sigma_\eta}}\sigma_\eta^{2*}}}{\mathrm{e}^{\frac{1}{2V_{\sigma_\eta}}\sigma_\eta^{2}}}\right\}.$$

Similarly, we can sample $\sigma_{\eta_x}^2$ by first obtaining a candidate $\sigma_{\eta_x}^{2*} \sim \mathcal{IG}((T-1)/2, \sum_{t=1}^{T}\eta_{x,t}^2/2)$. Then, given the current $\sigma_{\eta_x}^2$, we accept the candidate with probability

$$\min\left\{1, \frac{\mathrm{e}^{\frac{1}{2V_{\sigma_{\eta_x}}}\sigma_{\eta_x}^{2*}}}{\mathrm{e}^{\frac{1}{2V_{\sigma_{\eta_x}}}\sigma_{\eta_x}^{2}}}\right\}.$$

Step 5. Next, we jointly sample $\boldsymbol{\delta} = (\boldsymbol{\zeta}', \boldsymbol{\tau}_0')'$ from its full conditional distribution. To that end, we write (A.5)– (A.6) as a regression with coefficient vector $\boldsymbol{\delta}$:

$$\mathbf{H}_{\boldsymbol{\varphi}}\boldsymbol{\tau} = \mathbf{X}_{\boldsymbol{\delta}}\boldsymbol{\delta} + \boldsymbol{\eta},$$

where

$$
\mathbf{X}_{\boldsymbol{\delta}} = \begin{pmatrix} & & 1+\varphi_\mu & -\varphi_\mu & 0 & 0 \\ & & 0 & 0 & 1+\varphi_{\mu_x} & -\varphi_{\mu_x} \\ & & -\varphi_\mu & 0 & 0 & 0 \\ \mathbf{d}_0 \otimes \begin{pmatrix} 1-\varphi_\mu & 0 \\ 0 & 1-\varphi_{\mu_x} \end{pmatrix} & \mathbf{d}_1 \otimes \begin{pmatrix} 1-\varphi_\mu & 0 \\ 0 & 1-\varphi_{\mu_x} \end{pmatrix} & 0 & 0 & -\varphi_{\mu_x} & 0 \\ & & 0 & 0 & 0 & 0 \\ & & \vdots & \vdots & \vdots & \vdots \\ & & 0 & 0 & 0 & 0 \end{pmatrix}.
$$

Then, by standard regression results, we have

$$
(\boldsymbol{\delta} \,|\, \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\sigma}^2) \sim \mathcal{N}(\widehat{\boldsymbol{\delta}}, \mathbf{K}_{\boldsymbol{\delta}}^{-1}),
$$

where

$$
\mathbf{K}_{\boldsymbol{\delta}} = \mathbf{V}_{\boldsymbol{\delta}}^{-1} + \mathbf{X}_{\boldsymbol{\delta}}'\boldsymbol{\Sigma}_{\boldsymbol{\eta}}^{-1}\mathbf{X}_{\boldsymbol{\delta}}, \quad \widehat{\boldsymbol{\delta}} = \mathbf{K}_{\boldsymbol{\delta}}^{-1}\left(\mathbf{V}_{\boldsymbol{\delta}}^{-1}\boldsymbol{\delta}_0 + \mathbf{X}_{\boldsymbol{\delta}}'\boldsymbol{\Sigma}_{\boldsymbol{\eta}}^{-1}\mathbf{H}_{\boldsymbol{\varphi}}\boldsymbol{\tau}\right),
$$

where $\mathbf{V}_{\boldsymbol{\delta}} = \mathrm{diag}(\mathbf{V}_{\boldsymbol{\zeta}}, \mathbf{V}_{\boldsymbol{\tau}0})$ and $\boldsymbol{\delta}_0 = (\boldsymbol{\zeta}_0', \boldsymbol{\tau}_{00}')'$.

A.6 A Monte Carlo Study

The section provides Monte Carlo results to assess the empirical performance of two Bayesian procedures to test the null hypothesis that $\gamma = 0$. The first approach constructs a 95% credible interval for γ, and the null hypothesis is rejected if the credible interval excludes 0. The second approach computes the Bayes factor against the null hypothesis via the Savage-Dickey density ratio $p(\gamma = 0)/p(\gamma = 0 \,|\, \mathbf{y})$. Following Kass and Raftery (1995), we interpret a value of Bayes factor larger than $\sqrt{10}$ as substantial evidence against the null hypothesis; a value less than $1/\sqrt{10}$ is viewed as substantial evidence in favor of the null hypothesis; and the test is indecisive for values in between.

We first generate 200 datasets from the the unobserved components model in (1)–(6) with $\gamma = 0$ and $T = 500$. The values of other parameter are set to be the same as the estimates reported in Table 1. Given each dataset, we then estimate the model and conduct the two hypothesis tests described above. The results are reported in Table 5. For the first hypothesis test based on the credible interval, it works well and has about the right size. For the test based on the Bayes factor, it rejects the null hypothesis $\gamma = 0$ for about 2.5% of the datasets, whereas it rejects the alternative hypothesis $\gamma \neq 0$ for about 8.5% of the datasets.[8] That is, when the null hypothesis is true, the Bayes factor favors the null hypothesis 3.4 times more frequently than the alternative $\gamma \neq 0$ (though it is indecisive for the majority of datasets.)

[Table 5 about here.]

Next, we repeat the Monte Carlo experiment but now we set $\gamma = 1$. In this case, the hypothesis test based on the credible interval rejects the null hypothesis for about 81% of the datasets. For the test based on the Bayes factor, it rejects the null hypothesis for about 84% of the datasets and it never rejects the alternative hypothesis $\gamma \neq 0$. Overall, these Monte Carlo results show that both hypothesis tests work reasonably well.

[8] Recall that the Bayes factor presents the weight of evidence for or against the null hypothesis, and is not designed to have a fixed rejection rate.

Table 2: ADF test: testing the null hypothesis of the presence of a unit root.

Test	Variable	Lags	Test Statistic	Critical Value	Reject Null
ADF	Log RPI	1	-1.6893	-3.4310	No
ADF	Log RPI	2	-1.7358	-3.4310	No
ADF	Log RPI	5	-1.8142	-3.4313	No
ADF	Log TFP	1	-2.5138	-3.4310	No
ADF	Log TFP	2	-2.4844	-3.4310	No
ADF	Log TFP	5	-2.3264	-3.4313	No

Note: ADF stands for Augmented Dickey Fuller test. RPI and TFP correspond to the relative price of investment and total factor productivity, respectively. In all cases, the model includes a constant and a time trend. The data series span from 1959.II to 2019.II.. The lag order is selected according to the SIC and HQ criteria.

Table 3: KPSS test: testing the null hypothesis of stationarity.

Test	Variable	Lags	Test Statistic	Critical Value	Reject Null
KPSS	Log RPI	1	2.9071	0.1460	Yes
KPSS	Log RPI	2	1.9482	0.1460	Yes
KPSS	Log RPI	5	0.9899	0.1460	Yes
KPSS	Log TFP	1	0.9144	0.1460	Yes
KPSS	Log TFP	2	0.6196	0.1460	Yes
KPSS	Log TFP	5	0.3238	0.1460	Yes

Note: RPI and TFP correspond to the relative price of investment and total factor productivity, respectively. In all cases, the model includes a constant and time trend. The data series span from 1959.II to 2019.II. The lag order is selected according to the SIC and HQ criteria.

Table 4: Johansen's trace test for co-integration between RPI and TFP.

Null	Alternative	Deterministic trend	Lags	p-value
$r = 0$	$r > 0$	Yes	1	0.0845
$r = 0$	$r > 0$	Yes	3	0.0975
$r = 0$	$r > 0$	Yes	7	0.0236

Note: The co-integration tests are performed on the logarithms of the relative price of investment and total factor productivity. The sample period is 1959.II to 2019.II. The variable r denotes the number of co-integrating vectors. RPI and TFP correspond to the relative price of investment and total factor productivity, respectively, and the model includes a constant and time trend. The lag order is selected according to the SIC and HQ criteria.

Table 5: Frequencies (%) of rejecting the null hypothesis $\gamma = 0$ from the two hypothesis tests: a 95% credible interval excluding 0 and a Bayes factor value larger than $\sqrt{10}$.

True value	CI excludes 0	BF $> \sqrt{10}$	indecisive	BF $< 1/\sqrt{10}$
$\gamma = 0$	0.04	0.025	0.89	0.085
$\gamma = 1$	0.81	0.84	0.16	0