

# Invariant Inference and Efficient Computation in the Static Factor Model

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## ABSTRACT

Factor models are used in a wide range of areas. Two issues with Bayesian versions of these models are a lack of invariance to ordering of and scaling of the variables and computational inefficiency. This paper develops invariant and efficient Bayesian methods for estimating static factor models. This approach leads to inference that does not depend upon the ordering or scaling of the variables, and we provide arguments to explain this invariance. Beginning from identified parameters which are subject to orthogonality restrictions, we use parameter expansions to obtain a specification with computationally convenient conditional posteriors. We show significant gains in computational efficiency. Identifying restrictions that are commonly employed result in interpretable factors or loadings and, using our approach, these can be imposed ex-post. This allows us to investigate several alternative identifying (non-invariant) schemes without the need to respecify and resample the model. We illustrate the methods with two macroeconomic datasets.

**Keywords:** Bayesian, Factor model, Invariance, Reduced rank regression.

## 1 Introduction

Factor models have proven useful in many areas including psychology, genomics, epidemiology, economics and finance and significant advances in computation using Bayesian approaches (for example, Geweke and Zhou (1996), Aguilar and West (2000) and Chib, Nardari, and Shephard (2006)) have made Bayesian analysis of such models feasible for a range of applications. Two problems that have hampered Bayesian inference in factor models are, first, the models are not invariant to different ordering of the variables (see, for example, Lopes and West (2004)) and, second, poor efficiency of computation algorithms (e.g., Chib *et al.* (2006)).

This paper makes a number of contributions. i) This paper presents an invariant specification. That is, the specification will result in inference that does not depend upon the ordering of the variables. ii) We use parameter expansions to develop an algorithm that is both easy to implement and computationally efficient. The resulting posteriors have relatively simple normal forms. Further, as with the extant non-invariant specifications, our specification is overparameterised. However, we follow the rules of Liu and Wu (1999) to ensure efficiency gains. It is not clear that extant specifications do follow these rules and this may explain to some degree the poor sampling. iii) Finally, we provide a formal explanation for why extant specifications are not invariant, that is, why the evidence in the model can change when the order of the variables changes. In doing so, we demonstrate that there is not an identification problem so much as a specification problem in these models.

Reordering of variables involves groups of transformations of the parameters in the model. We therefore use group theory to show why existing specifications are not invariant to reordering of the variables. Work to date considering invariance has taken one of two approaches.

The first approach attempts to resolve the issue by averaging over orderings (see for example Geweke (1996) and Frühwirth-Schnatter and Lopes (2010)). To estimate  $k$  we would need to estimate all orderings for all values of  $k$ . Averaging over orderings shows promise in small dimensional settings, but as applications often have many variables (sometimes hundreds), the number of potential orderings to average over increases into the trillions making an averaging approach computationally infeasible or at best challenging. For example, Forni, Giannone, Lippi and Reichlin (2009) investigate some 89 series and find there are between  $k = 12$  and  $k = 18$  factors. In this case, if we were to use a non-invariant Bayesian approach and average over all orderings we would need to average over more than 237 trillion for  $k = 12$  and 3 million trillion models for  $k = 18$ . It would seem more practical and feasible to only have one invariant model to consider for each  $k$ .

Another approach, therefore, is to develop a single model that does not depend upon the orderings. Examples of work taking this approach, besides our paper, are Bhattacharya and Dunson (2011), Aßmann, Boysen-Hogrefe, and Pape (2012) and Kaufmann and Schumacher (2012). Our approach differs from these in that we explicitly take the perspective of the factor model as a reduced rank regression model, such as in Bai and Ng (2002), and use previous work that utilizes the geometry of that model to develop an invariant model specification and inferential framework. Taking this perspective leads us to the view that, contrary to general belief, there is not an identification problem in the factor model but rather there is a problem with the specification used.

The invariant specification we propose uses a singular value decomposition as in Hoff (2007). This approach is related to the principal components specification commonly used in the frequentist literature (see, for example, Connor and Korajczyk (1986) and Bai and Ng (2002))

and efficient computation is achieved by combining this with an extension of the parameter expansion approach of Ghosh and Dunson (2009) in the static factor model and Koop, León-González and Strachan (2010 & 2012) in the vector error correction model and instrumental variables model. The parameter expansions are chosen to obtain a specification that is simple to implement, in fact simpler than standard extant specifications. A further benefit of this expanded specification is that the resulting sampler is efficient. This parameter expansion may be viewed as a generalization of the Ghosh and Dunson (2009) approach to computing factor models but with the added benefit of invariance.

In Section 2 we present the identified parameters in the invariant Bayesian specification of the static factor model and the priors for this model from a singular value decomposition (SVD) (contribution i). We then introduce the full parameter expansion using invariant transformations to obtain the prior for the ‘expanded’ model (contribution ii). We present the posterior, sampling algorithm and the posterior probability estimation for this ‘expanded’ model. In Section 3 we briefly outline the features of the static factor model and discuss relationships, in particular mappings, among existing identification schemes, two of which are popular non-invariant specifications used in the Bayesian literature and one invariant specification used in the frequentist literature. This discussion allows us to present the source of invariance (contributions iii, iv and v). Section 4 presents several applications including one small application to six exchange rates to demonstrate the effect of reordering and efficiency of the proposed sampling algorithm in this paper. Section 5 provides some concluding comments and potential extensions.

## 2 The invariant static factor model

In this section we present the invariant specification of the static factor model. In the model stacked over time, we will show that the product of the matrix of all factors and the loading matrix forms a reduced rank matrix. The row and column space of this matrix are identified (as are various norms) and there is a smooth relationship between these spaces and appropriate orthonormal frames which we take advantage of to achieve an invariant specification. The parameters identified under the likelihood are the elements of the singular value decomposition of a reduced rank matrix, and therefore some of the identified parameters are orthonormal  $k$ -frames that belong to the Stiefel manifold. As discussed in Section 3, most of the literature have attempted to estimate parameters which are discontinuous transformations of the identified parameters, and it is this discontinuity that causes lack of invariance. In contrast, we build on the theory of invariant measures and distributions on special manifolds (e.g. Amari (1985), Chikuse (2003)) and first specify priors directly on the identified parameters. In order to facilitate computations we then introduce non-identified parameters that allow us to define diffeomorphic transformations from the identified and non-identified parameters (the parameter expansions) to parameters with computationally convenient supports (Real space) and distributions (normal). Because the transformations are diffeomorphic the resulting approach preserves the invariant (order independent) inference, while allowing for a much more efficient algorithm for computations. The prior that we propose is defined in such a way that conditional posteriors belong to standard families and allow for simple and efficient computations. Although previous work by Hoff (2007) proposes an invariant approach that works directly with the identified parameters, we show in the Supplemental material A that thanks to the parameter expansions the efficiency gains

of our approach can be very large.

We can write the factor model as a reduced rank regression model for a  $1 \times n$  vector  $y_t \in \mathbb{R}^n$  with  $k < n$  factors as

$$y_t = f_t \Lambda + \varepsilon_t, \quad \varepsilon_t \sim N(0, \Sigma), \quad \text{for } t = 1, \dots, T \quad (1)$$

where  $f_t$  is a  $1 \times k$  vector,  $\Lambda$  is an  $k \times n$  matrix, and  $\varepsilon_t$  is a  $1 \times n$  vector with a diagonal covariance matrix denoted by  $\Sigma$ . By stacking observations equation (1) can be equivalently written as  $y = F\Lambda + \varepsilon = \Pi + \varepsilon$ , where  $y$  and  $\varepsilon$  are  $T \times n$  matrices with  $E(\text{vec}(\varepsilon)\text{vec}(\varepsilon)') = \Sigma \otimes I_T$ ,  $F$  is a  $T \times k$  matrix such that the matrix  $\Pi = F\Lambda$  has rank  $k$ . In this section we develop priors for  $\Lambda$  and  $F$  by beginning with the parameters that are identified under the likelihood and, via a series of parameter expansions, we obtain the prior and posterior for the expanded model.

The reduced rank model in (1) has the same structure as a one-mode analysis used in psychometrics (see, for example, Magnus and Neudecker (1988)) for which frequentist approaches to estimation are proposed. Bayesian inference in other reduced rank models, such as the cointegrating vector error correction model and the overidentified simultaneous equations model, has been extensively explored and this literature is informative on how to approach the analysis of this model. Bayesian approaches most relevant to this paper are Strachan and Inder (2004), Koop, Léon-González and Strachan (2010 & 2012).

Taking a singular value decomposition (SVD) of the reduced rank matrix  $F\Lambda$ , we have

$$F\Lambda = U_1 S_1 V_1' \quad (2)$$

$$U_1 \in V_{k,T} \quad V_1 \in V_{k,n}$$

$$S_1 = \text{diag}(s_1, s_2, \dots, s_k)$$

where  $s_i > s_{i+1} > 0$  for all  $i$  and  $V_{m,n}$  denotes the Stiefel manifold such that  $V_{m,n} = \{H(n \times m); H'H = I_m\}$  (for discussion, see Muirhead (1982)). All of the parameters  $U_1$ ,  $S_1$  and  $V_1$  are identified up to sign and have, respectively,  $Tk - \frac{k(k+1)}{2}$ ,  $k$  and  $nk - \frac{k(k+1)}{2}$  free elements.

In what follows, we define the trace of a square matrix  $A$  as  $tr(A)$ . We specify priors for  $U_1$ ,  $S_1$  and  $V_1$  with the form

$$\frac{f(S_1, V_1) (dS_1) (U_1' dU_1) (V_1' dV_1) c_O}{c_N c_U}$$

$$f(S_1, V_1) \propto \exp \left\{ -\frac{c_\lambda}{2} tr(V_1' \underline{M}^{-1} V_1 S_1^2) \right\} 2^{-k} |S_1|^{n-k} \prod_{i < j}^k (s_i^2 - s_j^2),$$

$$\frac{c_N}{c_O} = \int f(S_1, V_1) (dS_1) (V_1' dV_1), \quad c_U = \int (U_1' dU_1) = \frac{2^k \pi^{\frac{Tk}{2}}}{\Gamma_k\left(\frac{T}{2}\right)}$$

$$c_O = \int (C' dC) = \frac{2^k \pi^{\frac{k^2}{2}}}{\Gamma_k\left(\frac{k}{2}\right)}, \quad \Gamma_k\left(\frac{m}{2}\right) = \pi^{k(k-1)/2} \prod_{i=1}^k \Gamma\left[\frac{m-i+1}{2}\right],$$

where  $\underline{M}$  is a matrix that can be fixed equal to the identity matrix  $I_n$  for a prior that is invariant to ordering only, or equal to  $\Sigma$  for a prior that is invariant to both ordering and scale transformation. The prior for  $U_1$  is uniform on the Stiefel manifold (for further discussion see James (1954)) and the prior for  $V_1$  is uniform also when  $\underline{M} = I_n$  but not when  $\underline{M} = \Sigma$ . We give an explicit expression for  $c_N$  below. The diagonal elements of the matrix  $S_1$  have a ‘standard’ prior which we will show implies that, marginally on  $F$ ,  $\Lambda$  follows a Matrix variate t-distribution (Gupta and Nagar, 2000, p. 134) with degrees of freedom equal to  $(T + 1 - k - n)$ , zero mean and  $var(vec(\Lambda)) = \underline{M} \otimes I_k \frac{1}{c_\lambda} \frac{1}{(T-n-k-1)}$ . The priors for  $U_1$ ,  $S_1$  and  $V_1$  are all proper. The term  $c_\lambda$  is included to permit shrinkage of  $\Lambda$  towards zero or a more diffuse prior if desired.



## 2.1 Parameter expansions

We introduce the nonidentified parameters by two parameter expansions. From the first expansion, we obtain a normal form for the loading matrix and the second expansion results in a normal prior for the factors. These expansions do not affect the proper, independent priors for  $U$ ,  $S$  and  $V$ , uniform priors for  $U$  and  $V$ , and the standard prior for  $S$ .

Map from the SVD parameters  $(U_1, S_1, V_1)$  to the expanded parameters  $(U_1, \Lambda^*)$  by introducing the orthogonal matrix  $C \in O(k)$  via the transformation

$$\begin{aligned} U_1 S_1 V_1' &= U_1 C' C S_1 V_1' = U \Lambda^* \\ U_1 C' &= U, \quad C S_1 V_1' = \Lambda^*. \end{aligned} \tag{3}$$

The expanding parameter  $C$  is given a uniform distribution on  $O(k) : (C' dC)$ . The transformation  $\Lambda^* = C S_1 V_1'$  is a singular value decomposition of  $\Lambda^*$  such that Jacobian of this transformation can be calculated using results in, for example, James (1954, p.71). The transformation of measures for (3) is  $(U_1' dU_1) = (U' dU)$ , such that the prior now becomes

$$\frac{f(S_1) (dS_1) (U_1' dU_1) (V_1' dV_1) (C' dC)}{c_N c_U} = \frac{p^*(\Lambda^*) (d\Lambda^*) (U' dU)}{c_N c_U}$$

$$p^*(\Lambda^*) = \exp \left\{ -\frac{c_\lambda}{2} \text{tr} (\underline{M}^{-1} \Lambda^* \Lambda^*) \right\},$$

$$c_N = \int p^*(\Lambda^*) (d\Lambda^*) = \int f(S_1) (dS_1) (V_1' dV_1) (C' dC) = \left( \frac{2\pi}{c_\lambda} \right)^{\frac{nk}{2}} |\underline{M}|^{\frac{k}{2}},$$

and so  $\Lambda^* = C S_1 V_1'$  has a normal prior distribution such that  $p^*(\Lambda^*)$  has a form proportional to the density of a zero mean normal distribution with covariance matrix  $\frac{1}{c_\lambda} (\underline{M} \otimes I_k)$ . The semi-orthogonal matrix  $U$  has a uniform distribution over  $V_{k,T}$ . In the new parameterization,  $F\Lambda = U\Lambda^*$ , the matrix  $\Lambda^*$  is has a ‘nice’ form and prior but  $U$  is restricted to be semi-orthogonal, and so it would be difficult to obtain draws from the posterior.

To give the parameters a more computationally convenient form and prior distributions, we transform from  $(U, \Lambda^*)$  to  $(F, \Lambda)$  via the second parameter expansion. Introduce the  $k \times k$  rank  $k$  matrix  $\kappa$  with  $\frac{k(k+1)}{2}$  free parameters.  $\kappa$  may be, for example, lower triangular or symmetric. This matrix is used to obtain the following transformations:

$$U\Lambda^* = U\kappa\kappa^{-1}\Lambda^* = F\Lambda$$

where  $F = U\kappa$  and  $\Lambda = \kappa^{-1}\Lambda^*$ . It is easier to work with the transformation  $A = \kappa'\kappa = F'F$  and write the Jacobian of the bijective transformation from  $(A, U, \Lambda^*)$  to  $(A, F, \Lambda)$  (e.g. Muirhead (1982), p. 58, 66) as

$$\frac{p^*(\Lambda^*) (dA) (U'dU) (d\Lambda^*)}{c_N c_U} = \frac{p(\Lambda, F) J(F) (d\Lambda) (dF)}{c_N c_U}$$

$$J(F) = 2^k |F'F|^{-(T-n-k-1)/2}, \quad p(\Lambda, F) = \exp \left\{ -\frac{c_\lambda}{2} \text{tr} (\underline{M}^{-1} \Lambda' F' F \Lambda) \right\}$$

Clearly the presence of the determinant  $|F'F|$  in the above Jacobian would complicate computation, particularly as we prefer to have a more convenient form such as a normal distribution for  $F$ . Fortunately, we are free to choose the distribution of  $A$  and so we let this matrix have a Wishart Distribution with degrees of freedom such that the prior for  $A$  is proportional to

$$\exp \left\{ -\frac{1}{2} \text{tr} (A) \right\} |A|^{(T-n-k-1)/2} = \exp \left\{ -\frac{1}{2} \text{tr} (F'F) \right\} |F'F|^{(T-n-k-1)/2}.$$

When we introduce this into the full prior we obtain the following expression of the measure

$$\frac{|A|^{(T-n-k-1)/2} \exp \left\{ -\frac{1}{2} \text{tr} (A) \right\} p^*(\Lambda^*) (dA) (U'dU) (d\Lambda^*)}{c_N c_U c_A} \tag{4}$$

$$= \exp \left\{ -\frac{1}{2} \text{tr} (F'F) \right\} |F'F|^{(T-n-k-1)/2} p(\Lambda, F) |F'F|^{-(T-n-k-1)/2} (d\Lambda) (dF) c$$

$$= \exp \left\{ -\frac{1}{2} \text{tr} (F'F) \right\} p(\Lambda, F) (d\Lambda) (dF) c$$

where  $c = \frac{2^k}{c_N c_U c_A}$  and define  $c_A$  as

$$c_A = \int |A|^{(T-n-k-1)/2} \exp \left\{ -\frac{1}{2} \text{tr}(A) \right\} (dA) = 2^{(T-n)k/2} \Gamma_k \left( \frac{T-n}{2} \right).$$

We can summarize the transformations used to this point as

$$\Pi = U_1 S_1 V_1' = U_1 C' C S_1 V_1' = U \Lambda^* = U \kappa \kappa^{-1} \Lambda^* = F \Lambda \quad (5)$$

The resulting joint prior distribution for  $F$  and  $\Lambda$  is given by

$$p(\Lambda, F) (d\Lambda) (dF) = \exp \left\{ -\frac{1}{2} \text{tr}(F'F) \right\} \exp \left\{ -\frac{c_\lambda}{2} \text{tr}(\underline{M}^{-1} \Lambda' F' F \Lambda) \right\} (d\Lambda) (dF) c. \quad (6)$$

Recall that are  $(\Lambda, F)$  are unrestricted matrices and note also that  $p(\Lambda, QF) = p(\Lambda, F)$  for any orthogonal matrix  $Q$ , which confirms that the prior for the space of  $U_1$  is uniform.

Integrating (6) with respect to  $F$  we get that the marginal prior for  $\Lambda$  is a matrix variate t-distribution (e.g. Gupta and Nagar, 2000, p. 134) with zero mean and  $\text{var}(\text{vec}(\Lambda)) = \underline{M} \otimes I_k \frac{1}{c_\lambda} \frac{1}{(T-n-k-1)}$ . The resulting conditional priors have convenient normal forms such

that they will be conjugate with the usual specification for the model for  $y$ . That is, the conditional prior for  $\lambda = \text{vec}(\Lambda) | F$  is normal with zero mean and covariance matrix  $\underline{V}_\lambda =$

$\underline{M} \frac{1}{c_\lambda} \otimes (F'F)^{-1}$ . The conditional prior for  $f = \text{vec}(F) | \Lambda$  is normal with zero mean and covariance matrix  $\underline{V}_F = [I_k + c_\lambda \Lambda \underline{M}^{-1} \Lambda']^{-1} \otimes I_T$ . For the chosen transformations and

distributions for the unidentified parameters,  $C$  and  $\kappa$ , we have results from Liu and Wu (1999) ensuring the sampler will converge. Specifically, the transformations we use form

locally compact groups and the priors for the expanding parameters correspond to Haar measures. Further, the expanding parameters are independent of the identified parameters.

Parameter expansions have been used in earlier work in factor models to produce more efficient and simple sampling schemes (see, for a recent example, Ghosh and Dunson, 2009)

and to accelerate the EM algorithm in factor models (Liu, Rubin and Wu, 1998, Ročková and George, 2015). The approach in this paper is an application of that developed in Liu (1994) and Liu and Wu (1999) and shares some of the features of Ghosh and Dunson (2009). However, a contribution of this paper that distinguishes it from this earlier work is to use parameter expansion to also achieve invariant inference. This builds upon earlier work on estimation of reduced rank models (Koop, León-González and Strachan (2010) & (2012)) which is natural as the factor model can be represented as a particular type of reduced rank regression model.

## 2.2 Posterior Computations

In this section we extend the model to allow exogenous variables, provide priors for the other parameters in the model and discuss approaches to computing  $k$ . The static factor model is often specified with  $m$  exogenous variables collected into the  $(T \times m)$  matrix  $X$ . After the parameter expansions in the previous section we obtained the matrix of factors  $F$  and the loading matrix  $\Lambda$ . The model can now be written as

$$Y = X\beta + F\Lambda + \varepsilon, \quad \varepsilon \sim N(0, \Sigma \otimes I_T) \quad (7)$$

$$f = \text{vec}(F) | \lambda \sim N(0, \underline{V}_F), \quad \lambda = \text{vec}(\Lambda) | f \sim N(0, \underline{V}_\lambda), \quad (8)$$

$$b = \text{vec}(\beta) \sim N\left(0, \Sigma \otimes (X'X)^{-1} \frac{1}{c_\beta}\right) \quad (9)$$

where  $\underline{V}_F = [I_k + c_\lambda \Lambda \underline{M}^{-1} \Lambda']^{-1} \otimes I_T$  and  $\underline{V}_\lambda = \underline{M} \frac{1}{c_\lambda} \otimes (F'F)^{-1}$ . We assume each diagonal element of  $\Sigma = \text{diag}\{\sigma_i^2\}$  has an inverse gamma prior

$$p(\sigma_i^2) \propto (\sigma_i^2)^{-\frac{\nu+2}{2}} \exp\left\{-\frac{\nu}{2\sigma_i^2 \underline{\mu}_i}\right\} (d\sigma_i^2)$$

such that

$$p(\Sigma) \propto |\Sigma|^{-\frac{\nu+2}{2}} \exp \left\{ -\frac{\nu}{2} \text{tr}(\Sigma^{-1}\underline{\Omega}) \right\} (d\Sigma) \quad (10)$$

where  $\underline{\Omega} = \text{diag} \left\{ \frac{1}{\underline{\mu}_1}, \frac{1}{\underline{\mu}_2}, \dots, \frac{1}{\underline{\mu}_n} \right\}$ . To preserve scale invariance requires that the prior for  $\Sigma$  also be scale invariant. One option is to use the Jeffrey's prior and set  $\underline{\nu} = 0$  in (10). Another approach is to set each  $\underline{\mu}_i$  to be some function of the scale of the data. For example, we could set  $\underline{\mu}_i = \frac{1}{\hat{\sigma}_i^2}$  where  $\hat{\sigma}_i^2$  is the sample variance of  $y_i$ . By setting  $\underline{\Omega} = \text{diag} \{ \hat{\sigma}_1^2, \hat{\sigma}_2^2, \dots, \hat{\sigma}_n^2 \}$ , the prior in (10) permits both of these options. The conditional posteriors  $f|\lambda, b, \Sigma$ ,  $\lambda|f, b, \Sigma$  and  $b|f, \lambda, \Sigma$  are normal while the elements on the diagonal of  $\Sigma|f, \lambda, b$  are inverse gamma distributions. The precise form of these conditional posteriors are given in the Appendix.

One important question in factor models is the number of factors,  $k$ . The model with no factors occurs at the point  $\Lambda = 0$  and at this point the factors are excluded from the likelihood. Therefore we are able to use the Savage-Dickey density ratio (SDDR) to compute the Bayes factors for  $k = 0$  to  $k = k^*$ ,  $B_{0,k^*}$ , as (Verdinelli and Wasserman (1995)):

$$B_{0,k^*} = \frac{m_0}{m_{k^*}} = \frac{p(\Lambda = 0|y)}{p(\Lambda = 0)} c_{VW}$$

where  $m_{k^*}$  is the marginal likelihood for the model with  $k^*$  factors,  $p(\Lambda = 0|y)$  is the marginal posterior of  $\Lambda$  at the point  $\Lambda = 0$ ,  $p(\Lambda = 0)$  is the marginal prior for  $\Lambda$  evaluated at the same point, and  $c_{VW}$  is the correction factor proposed by Verdinelli and Wasserman (1995).

When  $\underline{M} = I_n$  the correction factor  $c_{VW}$  is equal to one, and the marginal prior of  $\Lambda$  is a matrix variate t-distribution so that the value of the prior ordinate can be calculated as  $p(\Lambda = 0) = c(2\pi)^{Tk/2}$ , where  $c$  was defined next to expression (4). When  $\underline{M} = \Sigma$ , the conditional prior of  $\Sigma$  given  $\Lambda = 0$  depends on  $k$ , and therefore the correction factor  $c_{VW}$  becomes different from one. As shown in the Supplemental material B, when  $\underline{M} = \Sigma$  the

ratio  $c_{VW}/p(\Lambda = 0)$  is given by:

$$\frac{c_{VW}}{p(\Lambda = 0)} = \left(\frac{\pi}{c_\lambda}\right)^{nk/2} \frac{\Gamma_k\left(\frac{T-n}{2}\right)}{\Gamma_k\left(\frac{T}{2}\right)} \left(\frac{\Gamma\left(\frac{\nu+T}{2}\right)}{\Gamma\left(\frac{\nu+k+T}{2}\right)}\right)^n \prod_{i=1}^n \left(\frac{\tilde{h}_{ii}}{2}\right)^{\frac{k}{2}} \quad (11)$$

where  $\tilde{h}_{ii}$  is the  $i^{th}$  diagonal element of  $\tilde{H} = \underline{\nu}(\underline{\Omega}) + (Y - X\hat{\beta})'(Y - X\hat{\beta})$ , and  $\hat{\beta} = (1 - \sqrt{c_\beta/(1 + c_\beta)})(X'X)^{-1}X'Y$ . With a sequence of  $G$  draws from the posterior, we can compute the conditional posterior  $p(\Lambda|\beta, F, \Sigma, y)$  at  $\Lambda = 0$  to estimate the required ratio as:

$$\hat{B}_{0,k^*} = \frac{\frac{1}{G} \sum_{i=1}^G p\left(\Lambda = 0 | \beta^{(i)}, F^{(i)}, \Sigma^{(i)}, y\right)}{p(\Lambda = 0)} c_{VW},$$

where  $i = 1, \dots, G$  indicates the draws from the posterior.

Note that we are able to use the SDDR because the point  $\Lambda = 0$  belongs to the parameter space. The SDDR however cannot be used in the context of most previous literature because the identifying restrictions often imposed on  $\Lambda$  imply that  $\Lambda = 0$  is no longer a point in the parameter space. Conversely, although the method of Chib (1995) can be used to calculate marginal likelihoods in the non-invariant specifications, it is less suited to our specification because the accuracy of the method relies on being able to estimate the posterior density accurately at a point of high posterior density. This task is slightly more difficult in our context because we have introduced non-identified parameters, which makes the augmented posterior density more disperse around the mode. However, the SDDR method has the advantage that it does not require further calculations beyond the basic estimation algorithm. The accuracy of both methods to calculate marginal likelihoods decreases with the dimension of  $y$  and the number of factors. In such situations one can use alternative methods such as that of Chan and Eisenstat (2015) or calculate predictive likelihoods (Geweke and Amisano, 2010), as we illustrate in the empirical applications of Section 4.

In the following section we provide the technical details for the several contributions of this

paper. The reader who prefers not to read the technical details in Section 3 and interested only in applying the approach may prefer to skip to Section 4.

### **3 An explanation for non-invariance with discussion**

In this section we provide an explanation for non-invariance (contribution iii from the introduction). We provide an informal explanation for the invariance followed by a theorem, the proof for which we leave to the paper’s Supplemental material C. To support this discussion, we describe existing invariant specifications closely related to the one we propose. We then outline some standard non-invariant identification schemes that have been used in Bayesian analysis of factor models to develop a more formal explanation of the source of the non-invariance. The identification issues with this model are well understood (see, for example, Lopes and West (2004), Geweke and Zhou (1996) and Anderson and Rubin (1956)). We do not repeat these discussions here but focus upon the question of invariance.

The cause of the lack of invariance can be explained informally as follows. In the support for the loading matrix,  $\Lambda$ , there are points where subsets of columns form singular matrices. The collections of these points form manifolds in the support. Changing the order of the variables in  $y_t$  induces a transformation,  $g$ , of the loading matrix. These points of singularity in the support pose no problems for the transformation for the invariant specification as  $g$  is smooth, continuous and homoemorphic in this case. From group theory, we know these conditions ensure invariance. For the non-invariant specifications, the transformation is discontinuous, and therefore not bijective or homeomorphic, thus losing invariance. We demonstrate the importance of this discontinuity for inference on the number of factors with a simulation in the Supplemental materials D and E.

The geometric object of interest is the space spanned by the rows of the loading matrix,  $\mathfrak{p} = \text{space}(\Lambda')$ . The space  $\mathfrak{p}$  is a  $k$ -dimensional plane in  $n$ -space. The collection of all  $k$ -planes in  $n$ -space is the Grassman manifold,  $G_{n-k,n}$ . For the invariant specification, the support of  $\mathfrak{p}$  is the Grassman manifold under all orderings and the transformation of  $\Lambda$  with reordering implies a bijective mapping. With non-invariant specifications the support of  $\mathfrak{p}$  differs for each ordering. Let  $i$  denote a particular ordering of the variables. The ordering  $i$  implies that a submatrix made up of  $k$  columns of  $\Lambda$ ,  $\Lambda_i$ , must have full rank. Denote the points where  $|\Lambda_i| = 0$  by  $S_i$  then the support of  $\mathfrak{p}$  under order  $i$  is  $\mathfrak{L}_i = G_{n-k,n} - S_i$ . Under another ordering,  $j$ , the support  $\mathfrak{L}_j$  for  $\mathfrak{p}$  may intersect with the points  $S_i$  and the transformation  $g$  from  $\mathfrak{L}_i$  to  $\mathfrak{L}_j$  will not be surjective. The topology of  $\mathfrak{p}$  is not preserved and at the points  $|\Lambda_j| = 0$  the transformation  $g$  will have a discontinuity.

**Theorem 1 (Discontinuity)** *The transformation from  $\mathfrak{L}_i$  to  $\mathfrak{L}_j$  from reordering the variables has a discontinuity at  $|\Lambda_j| = 0$ .*

**Proof.** See Supplemental material C. ■

The restrictions on the support of  $\mathfrak{p}$  suggest another reason the invariant specification may be attractive. It is not possible to map from the non-invariant specifications to the invariant specification with the unrestricted support. However, it is possible to transform from the invariant specification to the non-invariant specifications. There may be some reason to prefer a particular non-invariant specification, or we may be interested in two or more alternative non-invariant specifications. In this case, we can estimate the model once with the invariant specification and post-process the results to obtain estimates of the non-invariant models.



## 4 Empirical applications

In this section we present two empirical examples to demonstrate the effect of lack of invariance of other approaches and that our approach achieves invariance. We also present evidence on the efficiency of the sampling algorithm.

### 4.1 Currency Exchange Rates

We use data on international currency exchange rates relative to U.S. dollar over a time period of 1045 business days beginning in January 2007 and ending in December 2010. The returns are computed as  $y_{it} = 100(p_{it}/p_{i,t-1} - 1)$ , where  $p_{it}$  denotes the daily closing spot rate. We first use six exchange rates and at the end of this example we expand the dataset to 20 exchange rates. The six series we analyze first are the Australian Dollar (AUD), Euro (EURO), South Korean Won (KRW), Japanese Yen (JPY), Canadian Dollar (CAD), and British Pound (GBP). These represent some of the most heavily traded currencies over the period. The returns are plotted in Figure 1.

We fit the data using the static factor model in (1) where  $y_t$  is the vector of six observed currency returns,  $f_t \sim iidN(0, I_k)$  is a  $1 \times k$  vector of unobserved factors,  $\Lambda$  is an  $k \times n$  matrix of factor loadings. We first impose the identification assumption that  $\Lambda$  is upper triangular with positive diagonal elements. We then compute the marginal likelihoods for four models. Using the ordering (AUD, EUR, KRW, JPY, CAD, GBP), we compute a single-factor model and then a two-factor model. Next, with the ordering (AUD, KRW, EUR, JPY, CAD, GBP), we again compute a single-factor model and then a two-factor model.

The log marginal likelihoods for the invariant specification are computed using the SDDR described earlier, and for the non-invariant specifications they are computed via the method

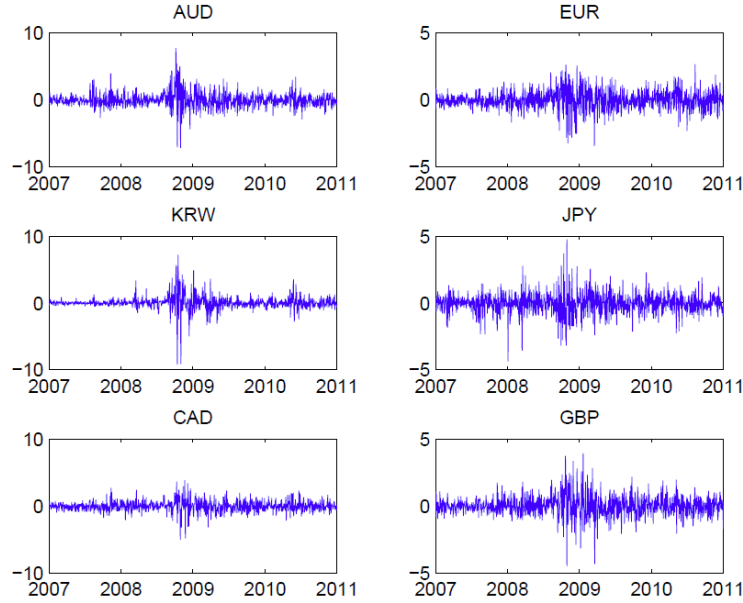


Figure 1: Daily returns of the six currencies.

of Chib (1995) using 100 parallel chains each of length 50000. The results are reported in Table 1. As the two marginal likelihoods for the models with one factor are almost the same (i.e., ordering made no difference) we only report one of these. The computed marginal likelihoods for the two-factors provide striking evidence of the effect of reordering. The log marginal likelihoods differ by about 142.

Under the ordering (AUD, KRW, EUR, JPY, CAD, GBP) there is a very strong preference for the two factor model over the one factor model with a log Bayes factor of -63.6. However, under the ordering (AUD, EUR, KRW, JPY, CAD, GBP), there is a very strong preference for the one factor model with the log Bayes factor of 78.3. The reordering of the variables has shifted the evidence on the number of factors in the opposite direction. The invariant specification selects a model with two factors over the one factor model with a log Bayes factor of 645 (for  $\underline{M} = \Sigma$ ). The evidence for two factors is therefore overwhelming.

Table 1: Log marginal likelihoods and the corresponding numerical standard errors for the competing models.

Model	log marginal likelihood	numerical standard error
1-factor (AUD, KRW, EUR, JPY, CAD, GBP)	-7572.9	3.40
2-factor (AUD, EUR, KRW, JPY, CAD, GBP)	-7636.5	5.64
2-factor (AUD, KRW, EUR, JPY, CAD, GBP)	-7494.6	1.16

Figure 2 reports the inefficiency factors for blocks of parameters from the model estimated with the Geweke and Zhou specification ( $GZ$ ) and the parameter expanded invariant model ( $PX$ ). The inefficiency measures give an estimate of the number of draws needed to have as much information about the posterior as we would obtain from one independent draw. The smaller the inefficiency factor the better is the sampler. It is clear that the parameters are generally much more efficiently estimated using the expanded parameter specification. These results are consistent with those found in Ghosh and Dunson (2009). The distributions of both the loading matrix,  $\Lambda$ , and the factors,  $F$ , are less disperse and lower for the parameter expanded model, but this is also true for the idiosyncratic variances,  $\Sigma$ , and the exogenous variables coefficients,  $\beta$ .

To illustrate how the method performs in higher dimensions, we add to  $y$  14 additional currencies, so that  $n = 20$ . In order to control for possible serial correlation, we use the model with exogenous variables in (7) with the intercept term and one lag of  $y_t$  in  $X_t$ , so that  $m = 21$  and  $T = 1043$ . Because the SDDR has a high variance in high dimensions, we calculate the marginal likelihood using the adaptive importance sampling method proposed in Chan and

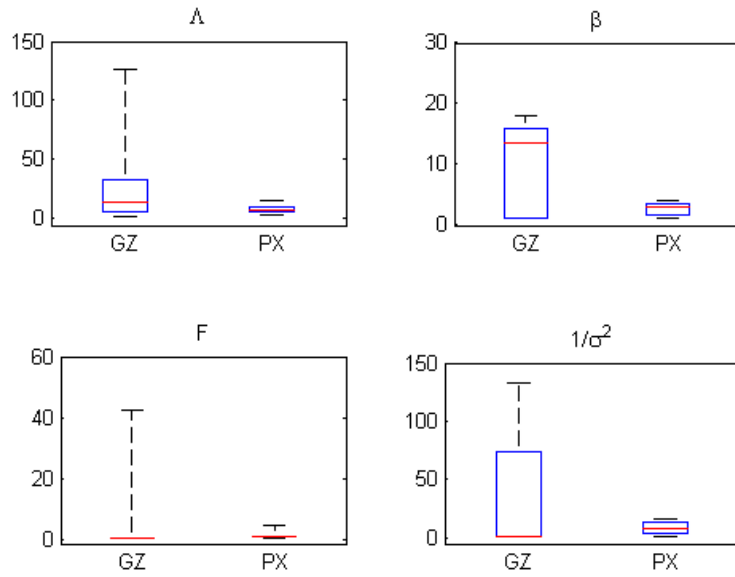


Figure 2: Box and whisker plots of inefficiency factors for blocks of parameters:  $(\Lambda, \beta, F, \Sigma^{-1}$  (denoted as  $1/\sigma^2$ )). The models were estimated using Gibbs sampling. *GZ* refers to the model with the upper triangular  $\Lambda_1$  with positive elements on the main diagonal, and *PX* refers to the parameter expansion of the invariant specification.

Table 2: Logs of Marginal Likelihood with 20 exchange rates and Numerical Standard Errors.

$k$	1	2	3	4	5	6	7
Log ML	-19092	-18516	-18397	-18335	-18324	-18367	-18412
NSE	0.08	0.15	0.16	1.24	3.02	4.97	3.86

Eisenstat (2015). Specifically, an importance sampling density for  $(\beta, \Lambda, \Sigma)$  is first obtained by approximating the joint posterior density. Then we compute the importance sampling estimate using the integrated likelihood, that is, the conditional density of the data marginal of  $F$ , which is available analytically. For each factor model, 10000 posterior draws are used to compute the importance sampling density. Then, 100000 importance sampling draws are obtained to calculate the marginal likelihood. Table 2 reports log marginal likelihoods and numerical standard errors, showing that the marginal likelihood improves up to the model with 5 factors and then decreases, indicating that the model with 5 factors is the most adequate.

## 4.2 The Number of Factors Driving US Macroeconomic Indicators

The dataset is obtained from Stock and Watson (2009), which consists of 190 quarterly observations from 1959Q3 to 2006Q4 on  $n = 109$  macroeconomic variables. Stock and Watson (2009) provide a detailed list of the data and its transformation in terms of logs and differencing. The dataset includes variables on GDP, industrial production, capacity utilization, purchasing manager's indices, labor force statistics, housing starts, consumer prices, commodity prices, average hourly earnings, productivity, interest rates, yield spreads, exchange rates, stock prices, money bases, business loans, consumer credit and consumer expectations. The data are first standardized.

Table 3: Sums of predictive likelihoods in the  $k$ -factor model.

$k$	1	2	3	4	5	6	7
	-5545.0	-5493.0	-5425.2	-5400.1	-5388.0	-5335.3	-5360.9

In Table 3 we report the sums of log predictive likelihoods for the last 10 years using a  $k$ -factor model, with  $k = 1, \dots, 7$ . Although Stock and Watson (2009) used a dynamic factor model, here for simplicity we use the static factor model of Section 2. Predictive likelihoods are the one-step ahead predictive density evaluated at the realized outcome (see Geweke and Amisano 2010 or Geweke, 1996). Due to the large dimension of  $y_t$ , predictive likelihoods can be calculated more accurately than the SDDR, especially when parallel computing is available. The best performing model is the one with  $k = 6$ , which gains as much as 25 points in the log scale with respect to the second best model ( $k = 7$ ). This result is within the range suggested by Stock and Watson (2009), who used the method of Bai and Ng (2002) and found the rank to be between 2 and 10 depending on the criterion used.

## 5 Concluding remarks

In this paper, we propose a specification for the static factor model that requires no ordering restrictions and so the choice of number of factors cannot depend upon the chosen ordering. By augmenting the posterior with a number of unidentified parameters with appropriate priors, the model can be computed using standard distributions and the draws are relatively efficient.

The specification we propose nests many of the existing and popular specifications used in factor analysis. Thus each of these specifications are attainable directly from the output from estimating our specification.

Although for convenience we have only considered the static factor model, this approach is readily extended to allow dynamics in the state equation. Such an extension would involve using an informative prior on the space  $sp(F)$  such as the orthogonal projective Gaussian distribution as used in Koop, Léon- González and Strachan (2011). This would involve transforming from  $F$  to  $F_c = RF$  where the matrix  $R$  captures the dynamics in  $F_c$ . For example, if the rows of  $F$  follow a random walk as  $f_{c,t} = f_{c,t-1} + f_t = \sum_{i=1}^t f_i$  then we would define  $R$  as

$$R = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Alternatively  $R = R(\rho)$  may be a function of parameters to allow richer dynamics such as in autoregressive processes. For example, an  $AR(1)$  state equation  $f_{c,t} = \rho f_{c,t} + f_t = \sum_{i=1}^t \rho^{t-i} f_i$  and so in which case we define  $R$  as

$$R(\rho) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \rho & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ \rho^{t-1} & \rho^{t-2} & \cdots & 1 \end{bmatrix}.$$

The full matrix of factors then becomes  $F_c = R(\rho)F$  which implies a prior for a process with a zero mean process with covariance matrix  $(I_k + c_\lambda \Lambda \Lambda')^{-1} \otimes (R(\rho)' R(\rho))^{-1}$ .

Another implication of the invariant specification, which we have exploited in this paper, is that we are able to compute the Bayes factors for the number of factors using the Savage-Dickey density ratio. This approach requires only the conditional posterior and the conditional prior for  $\Lambda$ . This greatly simplifies the computation of the posterior probabilities.

This approach cannot be used in the non-invariant specifications as they exclude the point  $\Lambda = 0$  from the support of the loading matrix parameter.

In computing the models in this paper, it became evident that there is a relationship between the computational efficiency and accuracy of marginal likelihood estimates, and the proximity of the posterior to the point of discontinuity. Models that are specified such that the posterior is invariant to reordering tend to have lower numerical standard errors. The accuracy of estimation of the marginal likelihood plays an important role in the confidence we have in the conclusions we make. This relationship is a topic of current research.

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## Appendix: Conditional Posterior Densities

The likelihood function can be written as

$$L \propto |\Sigma|^{-T/2} \exp \left[ -\frac{1}{2} \text{tr} \left( \Sigma^{-1} (Y - X\beta - F\Lambda)' (Y - X\beta - F\Lambda) \right) \right]$$

so that the conditional posteriors can be readily derived. First define

$$\begin{aligned} a &= \begin{bmatrix} \text{vec}(\beta) \\ \text{vec}(\Lambda) \end{bmatrix} = \begin{bmatrix} b \\ \lambda \end{bmatrix} \\ W &= [I_n \otimes X \quad I_n \otimes F] = (I_n \otimes \omega), \\ \omega &= [X \quad F], \end{aligned}$$

and  $f = \text{vec}(F)$ . Vectorizing the  $T \times n$  error matrix  $\varepsilon$  gives the useful linear forms for  $f$  and  $a$ . Let  $y = \text{vec}(Y)$ , then let  $x = (I_n \otimes X)$  and  $l = (\Lambda' \otimes I_T)$ , and define  $\tilde{y} = y - xb$  such

that we can write

$$\begin{aligned}
\text{vec}(Y - X\beta - F\Lambda) &= y - (I_n \otimes X)b - (\Lambda' \otimes I_T) f \\
&= \tilde{y} - lf \\
&= y - (I_n \otimes X)b - (I_k \otimes F)\lambda \\
&= y - Wa
\end{aligned}$$

As the vectors  $f$  and  $a$  have normal priors and enter the likelihood linearly (conditional on the other parameters) the conditional posteriors result from standard computations. Specifically, the conditional posteriors have the following forms:

$$f|\beta, \Lambda, \Sigma, Y \sim N(\bar{f}, \bar{V}_F), \quad a|F, \Sigma, Y \sim N(\bar{a}, \bar{V}_a), \quad \sigma_i^2|a, F \sim \bar{\mu}_i \chi_{\bar{\nu}}^{-2}$$

where

$$\begin{aligned}
\bar{f} &= \bar{V}_F (\Lambda \Sigma^{-1} \otimes I_T) (y - xb) = \text{vec} \left( (Y - X\beta) \Sigma^{-1} \Lambda' [\Lambda (\Sigma^{-1} + c_\lambda \underline{M}^{-1}) \Lambda' + I_k]^{-1} \right), \\
\bar{V}_F &= [\Lambda (\Sigma^{-1} + c_\lambda \underline{M}^{-1}) \Lambda' + I_k]^{-1} \otimes I_T, \\
\bar{a} &= \bar{V}_a W' (\Sigma^{-1} \otimes I_T) y = \bar{V}_a \begin{pmatrix} \text{vec}(X'Y\Sigma^{-1}) \\ \text{vec}(F'Y\Sigma^{-1}) \end{pmatrix}, \\
\bar{V}_a &= [W' (\Sigma^{-1} \otimes I_T) W + \underline{V}_a^{-1}]^{-1} = \begin{bmatrix} \Sigma^{-1} \otimes X'X & \Sigma^{-1} \otimes X'F \\ \Sigma^{-1} \otimes F'X & \Sigma^{-1} \otimes F'F \end{bmatrix} + \underline{V}_a^{-1} \\
\underline{V}_a &= \begin{bmatrix} \Sigma \otimes (X'X c_\beta)^{-1} & 0 \\ 0 & \underline{M} \otimes (c_\lambda F'F)^{-1} \end{bmatrix} \\
y &= \text{vec}(Y) \\
\bar{\mu}_i &= h_{ii}
\end{aligned}$$

where  $\bar{\nu} = T + m + \underline{\nu}$ , when  $\underline{M} = I_n$  and  $\bar{\nu} = T + k + m + \underline{\nu}$ , when  $\underline{M} = \Sigma$ . In the above expression,  $h_{ii}$  is the  $i^{\text{th}}$  diagonal element of  $H$ , and  $H = \varepsilon' \varepsilon + \underline{\nu} \underline{\Omega} + c_\beta \beta' X' X \beta$  when  $\underline{M} = I_n$

and  $H = \varepsilon'\varepsilon + \underline{\nu}\underline{\Omega} + c_\beta\beta'X'X\beta + c_\lambda\Lambda'F'F\Lambda$  when  $\underline{M} = \Sigma$ . The notation  $\bar{\mu}_i\chi_{\bar{\nu}}^{-2}$  refers to  $\bar{\mu}_i$  times an inverse chi-squared with  $\bar{\nu}$  degrees of freedom (e.g. Lee (2012, p.377)).