

# Asymptotic Trimming for Importance Sampling Estimators with Infinite Variance

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March 2017

## Abstract

Importance sampling is a popular Monte Carlo method used in a variety of areas in econometrics. When the variance of the importance sampling estimator is infinite, the central limit theorem does not apply and estimates tend to be erratic even when the simulation size is large. We consider asymptotic trimming in such a setting. Specifically, we propose a bias-corrected tail-trimmed estimator such that it is consistent and has finite variance. We show that the proposed estimator is asymptotically normal, and has good finite-sample properties in a Monte Carlo study.

Keywords: simulated maximum likelihood, bias correction, stochastic volatility, importance sampling

JEL classification: C11, C32, C52

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\*Joshua Chan would like to acknowledge financial support by the Australian Research Council via a Discovery Early Career Researcher Award (DE150100795).

# 1 Introduction

Importance sampling is widely used in econometrics to estimate integrals that do not have closed-form solutions. This Monte Carlo technique is especially important in maximum likelihood estimation of latent variable models—such as stochastic volatility and discrete choice models—where it is used to approximate intractable likelihood functions. Examples of maximum likelihood estimators computed using importance sampling for stochastic volatility models include [Sandmann and Koopman \(1998\)](#) and [Koopman and Hol Uspensky \(2002\)](#); for discrete choice models, see, e.g., [Geweke, Keane and Runkle \(1994\)](#) and [Hajivassiliou and McFadden \(1998\)](#). In the Bayesian literature, importance sampling is used in a variety of areas, including model comparison (e.g., [Frühwirth-Schnatter, 1995](#); [Frühwirth-Schnatter and Wagner, 2008](#); [Chan and Eisenstat, 2015](#)) and estimation using particle Markov chain Monte Carlo methods (e.g., [Pitt et al., 2012](#); [Tran et al., 2014](#)).

In his seminal paper, [Geweke \(1989\)](#) cautions that importance sampling should only be used when one can ensure the variance of estimator is finite. This is because when this finite variance condition fails, the central limit theorem does not apply, and the importance sampling estimator converges slower than the usual parametric rate. Even though in principle the estimator remains consistent, in practice it can be strongly biased and erratic even when the simulation size is huge. Despite these warnings, practitioners often ignore to check the finite variance condition, as it is challenging to verify in high-dimensional settings.

[Koopman, Shephard and Creal \(2009\)](#) make an important contribution by proposing a test to assess the validity of this finite variance assumption. However, it is not clear how one should proceed when the finite variance condition fails. This paper proposes a way forward in such settings. We ask if the original infinite-variance estimator can be modified such that the new estimator is consistent, and more importantly, asymptotically normal.

To that end, we consider asymptotic trimming (see, e.g., [Hill, 2010, 2013](#)). Specifically, we trim the right tail of the importance sampling weights in a way that the tail-trimmed estimator converges to the estimand faster than the untrimmed estimator. Trimming large importance sampling weights obviously introduces bias, and we show that this bias dominates the variance asymptotically. To overcome this problem, we propose a biased-corrected tail-trimmed estimator that has finite variance. We further prove that this estimator, after proper studentization, is asymptotically normal. Given this asymptotic

normality, standard machinery for inference can then be applied. For instance, the test statistic for a standard likelihood-ratio test will be asymptotically chi-squared distributed.

We demonstrate the good properties of the proposed estimator in a Monte Carlo study. We show that in cases when the variance of the original importance sampling estimator does not exist, its sampling distribution can be highly skewed. In contrast, the biased-corrected tail-trimmed estimator performs substantially better with an approximate normal distribution. We illustrate the proposed methodology with an application on fitting daily financial returns using a standard stochastic volatility model.

## 2 Importance Sampling

Importance sampling is a variance reduction technique that can be traced back to [Kahn and Marshall \(1953\)](#) and [Marshall \(1956\)](#). [Kloek and Van Dijk \(1978\)](#) appear to be the first application of importance sampling in econometrics. To define importance sampling, suppose we wish to evaluate the following integral that does not have an analytical expression:

$$\Psi = \int_{\mathcal{X}} H(\mathbf{x})f(\mathbf{x})d\mathbf{x} < \infty,$$

where  $\mathbf{x}$  is a  $k \times 1$  vector,  $H$  is a function mapping from  $\mathbb{R}^k$  to  $\mathbb{R}$ , and  $f$  is a density function with support  $\mathcal{X} \subset \mathbb{R}^k$ . Let  $g$  be another density function that *dominates*  $Hf$ , i.e.,  $g(\mathbf{x}) = 0$  implies that  $H(\mathbf{x})f(\mathbf{x}) = 0$ . Then,  $\Psi$  can be rewritten as

$$\Psi = \int_{\mathcal{X}} H(\mathbf{x})\frac{f(\mathbf{x})}{g(\mathbf{x})}g(\mathbf{x})d\mathbf{x}.$$

Therefore, we can estimate  $\Psi$  by the *importance sampling estimator*:

$$\widehat{\Psi}_n = \frac{1}{n} \sum_{i=1}^n \frac{H(\mathbf{X}_i)f(\mathbf{X}_i)}{g(\mathbf{X}_i)}, \tag{1}$$

where  $\mathbf{X}_1, \dots, \mathbf{X}_n$  are independent draws from the density  $g$ , which is often called the *importance sampling density*.

It is easy to see that  $\mathbb{E}_g \widehat{\Psi}_n = \Psi$ , where we make it explicit that the expectation is taken with respect to  $g$ . In addition,  $\widehat{\Psi}_n$  is a consistent estimator of  $\Psi$ . We refer the readers to [Kroese, Taimre and Botev \(2013\)](#) for a more detailed discussion of importance sampling. For later reference, we define  $W(\mathbf{X}) = H(\mathbf{X})f(\mathbf{X})/g(\mathbf{X})$  and write  $W(\mathbf{X}_i)$

hereafter simply as  $W_i$ .

## 2.1 Testing for Existence of Variance

One can establish the asymptotic distribution of the importance sampling estimator  $\widehat{\Psi}_n$  if  $W_i$  has a finite second moment. However, this is difficult to verify in practice. This problem has been recognized by [Monahan \(1993, 2001\)](#) and [Koopman, Shephard and Creal \(2009\)](#). In particular, [Monahan \(1993, 2001\)](#) provides a test for the existence of the second moment of  $W_i$  when the right tail of  $|W_i|$  satisfies the condition

$$1 - F(w) = c_w w^{-\alpha} [1 + O(w^{-\beta})] \quad (2)$$

as  $w \rightarrow \infty$ , where  $F$  is the cumulative distribution function of  $|W_i|$ . It follows from (2) that  $\mathbb{E}|W_i|^j < \infty$  if  $j < \alpha$  and  $\mathbb{E}|W_i|^j = \infty$  otherwise. Consequently, testing the existence of the second moment is equivalent to testing  $\alpha > 2$ .

Alternatively, [Koopman, Shephard and Creal \(2009\)](#) construct a test statistics from the maximum likelihood estimates of the generalized Pareto distribution with density

$$f(z; \xi, \eta) = \eta^{-1} \left(1 + \frac{z}{\xi\eta}\right)^{-(\xi+1)},$$

for  $z \in D(\xi, \eta) > 0, \eta > 0$ , where

$$D(\xi, \eta) = \begin{cases} [0, \infty), & \xi \geq 0, \\ [0, -\eta\xi], & \xi < 0. \end{cases}$$

The parameter  $\xi$  here plays the same role as  $\alpha$  above. In fact, the generalized Pareto satisfies equation (2), with  $\alpha = \xi$ ,  $\beta \rightarrow \infty$  when  $\xi > 0$ .<sup>1</sup>

In our context  $H$  is the likelihood function and therefore  $W_i = H(\mathbf{X}_i)f(\mathbf{X}_i)/g(\mathbf{X}_i) \geq 0$ . From here onwards we assume  $W_i$  is bounded from the left, i.e., there exists a positive number  $c_l$  that  $W_i > -c_l$ , and we only need to discuss the right tail of  $W_i$ .

Below we formally present the assumptions and results in [Monahan \(1993, 2001\)](#). These assumptions are also needed when we develop our bias-corrected tail-trimmed estimator

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<sup>1</sup>The result can be seen by a change of variable  $w = z + \xi\eta$ . Then the density function of  $w$  becomes  $f(w; \xi, \eta) = \eta^{-1}(\xi^{-1}w/\eta)^{-\xi-1}$ , for  $z \in \tilde{D}(\xi, \eta) > 0, \eta > 0$ , where  $\tilde{D}(\xi, \eta) = \begin{cases} [\xi\eta, \infty), & \xi \geq 0, \\ [\xi\eta, 0], & \xi < 0. \end{cases}$

in the following sections.

To motivate the test in [Monahan \(1993, 2001\)](#), we first consider the following technical conditions.

**Assumption 1** *There exists a positive number  $c_l$  such that  $W_i > -c_l$ . In addition, we assume  $\mathbb{E}W_i = \Psi < \infty$  and*

$$\mathbb{P}(W_i > w) = c_w w^{-\alpha} (1 + o(w^{-\beta}))$$

as  $w \rightarrow \infty$  and  $c_w, \alpha, \beta > 0$ .

**Assumption 2** *Let  $\{m_n\}$  be a sequence of positive integers such that  $m_n \rightarrow \infty$  and  $m_n = o(n^{2\beta/(2\beta+\alpha)})$  as  $n \rightarrow \infty$ .*

**Assumption 3** *The random vectors  $\{\mathbf{X}_i\}$  are independent and identically distributed.*

In [Assumption 1](#), condition  $W_i > -c_l$  restricts our attention to the right tail of  $W$ . In the general case where both tails of  $W$  are possibly heavy, we could apply the same analysis on  $|W|$ . Before we turn to the discussion of the rest part of the above assumptions, we consider the estimation of  $\alpha$ . To that end, we define the following quantities

$$\begin{aligned} \bar{F}(y) &= 1 - F(y), \\ b(y^{-1}) &= F^{\leftarrow}(1 - y), \end{aligned}$$

where  $F^{\leftarrow}(y) = \inf\{w : F(w) \geq y\}$ ,  $0 < y < 1$ . We define the sample order statistics  $W_{(i)}$  of  $W_1, \dots, W_n$  such that  $W_{(1)} \geq W_{(2)} \geq \dots \geq W_{(n)}$ .

[Hill \(1975\)](#) proposes the following estimator of  $\alpha$ :

$$\hat{\alpha}^{-1} = \frac{1}{m_n} \sum_{i=1}^{m_n} (\log W_{(i)} - \log W_{(m_n)}), \quad (3)$$

where  $m_n \rightarrow \infty$ ,  $m_n/n \rightarrow 0$  as  $n \rightarrow \infty$ .

[Assumptions 1–3](#) are one set of sufficient conditions for the asymptotic normality of  $\hat{\alpha}$ . The role of  $\beta$  in [Assumption 1](#) and [Assumption 2](#) is to ensure that  $\hat{\alpha}$  is consistent. For a more detailed discussion on the sufficient conditions for the asymptotic normality of  $\hat{\alpha}$ , see [Haeusler and Teugels \(1985\)](#).

Under Assumptions 1–3, the asymptotic normality of  $\hat{\alpha}$  can be established. We summarize this result in the following theorem.

**Theorem 2.1** *Under Assumptions 1–3, we have*

$$m_n^{1/2} (\hat{\alpha}^{-1} - \alpha^{-1}) / \alpha^{-1} \xrightarrow{d} \mathcal{N}(0, 1).$$

The above theorem is a direct result of Corollary 4 in [Haeusler and Teugels \(1985\)](#).

Here the convergence rate of  $\hat{\alpha}^{-1}$  is  $m_n^{1/2}$ . Assumption 2 restricts  $m_n$  such that  $m_n = o(n^{2\beta/(2\beta+\alpha)})$ . It implies that the larger the value  $\alpha$  has—corresponding to a thinner tail of  $W_i$ —the faster the convergence rate of  $\hat{\alpha}^{-1}$  can possibly achieve. The parameter  $\beta$  also plays an important role here. However, the estimation of  $\beta$  is unfortunately very hard; to the best of our knowledge, its estimation remains an open question in the literature.

Finally, [Monahan \(1993, 2001\)](#) proposes testing the existence of the second moment of  $W_i$  using the following test.

$$\mathbb{H}_0 : \alpha \geq 2 \text{ versus } \mathbb{H}_1 : \alpha < 2. \tag{4}$$

If  $\mathbb{H}_0$  is rejected, we then conclude that the variance of the importance sampling estimator is infinite.

We can perform the hypothesis test in (4) using Theorem 2.1. Specifically, we construct the test statistic  $m_n^{1/2} (\hat{\alpha}^{-1} - 2^{-1}) / \hat{\alpha}^{-1}$ . Since this test statistic is asymptotically normal, we calculate the P-value as  $1 - \Phi\left(m_n^{1/2} (\hat{\alpha}^{-1} - 2^{-1}) / \hat{\alpha}^{-1}\right)$ , where  $\Phi(\cdot)$  is the cumulative distribution function of the standard normal distribution.

## 2.2 A Tail-Trimmed Estimator

If  $\alpha$  is larger than 2, then the variance of the importance sampling estimator exists. It follows that the estimator converges to  $\Psi$  at the usual parametric rate and is asymptotically normal. Conversely, if the test rejects the null hypothesis that  $\alpha \geq 2$ , then the central limit theorem does not apply and one might need a prohibitively large simulation size to obtain a reliable estimate.

A natural question is: how can we proceed in the latter scenario? The literature has

focused on providing a test of  $\mathbb{H}_0$  and has not offered any solution if  $\mathbb{H}_0$  is rejected. We aim to take a first step in filling this gap. Specifically, we consider tail trimming, i.e., dropping some large values of  $W_i$ , in a such way that the resulting tail-trimmed estimator converges to  $\Psi$  faster than otherwise. More importantly, the modified estimator is asymptotically normal after proper studentization.

To that end, let  $\{k_n\}$  be an intermediate order sequence. That is,  $k_n \rightarrow \infty$  and  $k_n/n \rightarrow 0$  as  $n \rightarrow \infty$ . We define a sequence  $\{l_n\}$  as the  $(1 - k_n/n)$ -th quantile of  $W_i$ , i.e.,

$$\bar{F}(l_n) = \frac{k_n}{n}.$$

Next, we consider the following tail-trimmed estimator:

$$\widehat{\Psi}_n^* = \frac{1}{n} \sum_{i=1}^n W_i I(W_i < W_{(k_n)}), \quad (5)$$

where  $I(\cdot)$  is the indicator function. In other words, the estimator  $\widehat{\Psi}_n^*$  drops the largest  $k_n$  values of  $W_i$ .

Obviously,  $\widehat{\Psi}_n^*$  is biased downward. To discuss the bias and variance, let  $W_{n,i}^* = W_i I(W_i < l_n)$ . Then the bias  $B_n$  and the variance  $S_n^2$  of the tail-trimmed estimator are, respectively,

$$B_n = \mathbb{E}[W_i I(W_i \geq l_n)], \quad S_n^2 = \mathbb{E}([W_{n,i}^* - \mathbb{E}W_{n,i}^*]^2).$$

Below we derive the asymptotic distribution of  $\widehat{\Psi}_n^*$ . For that purpose, we consider the following technical assumption.

**Assumption 1'** *There exists a positive number  $c_l$  that  $W_i > -c_l$ . In addition, we assume  $\mathbb{E}W_i = \Psi < \infty$  and*

$$\mathbb{P}(W_i > w) = c_w w^{-\alpha} (1 + o(1))$$

*as  $w \rightarrow \infty$  and  $c_w, \alpha > 0$ .*

Assumption 1' imposes a tail restriction on  $W_i$  that is similar to, but weaker than, that in Assumption 1. We are able to do it because the stronger restriction in Assumption 1 is only used to ensure the asymptotic unbiasedness of  $\widehat{\alpha}^{-1}$ , which is not needed here.

The following theorem establishes the asymptotic normality of  $\widehat{\Psi}_n^*$ , even when  $W_i$  does not have a finite variance.

**Theorem 2.2** Suppose  $1 < \alpha \leq 2$ . Under Assumptions 3 and 1', and  $k_n \rightarrow \infty$ ,  $k_n/n \rightarrow 0$  as  $n \rightarrow \infty$ ,

$$n^{1/2} S_n^{-1} \left( \widehat{\Psi}_n^* - \Psi + B_n \right) \xrightarrow{d} \mathcal{N}(0, 1).$$

The proof is given in [Appendix A](#).

Even though the tail-trimmed estimator  $\widehat{\Psi}_n^*$  is asymptotically normal, the bias  $B_n$  dominates its distribution. To see that, we derive an approximate expression for  $B_n$  below. By Assumption 1', we have

$$\begin{aligned} B_n &= \mathbb{E}[W_i I(W_i \geq l_n)] \approx \int_0^{k_n/n} b(u^{-1}) du \\ &\approx c_w^{1/\alpha} \frac{\alpha}{\alpha - 1} \left( \frac{k_n}{n} \right)^{-1/\alpha+1} \\ &= \frac{\alpha}{\alpha - 1} \frac{k_n}{n} l_n. \end{aligned} \tag{6}$$

In the case when  $1 < \alpha < 2$ , we have  $S_n^2 \approx K(n/k_n)^{1/\alpha-1}$  for some constant  $K$ . Hence, the ratio of the bias and standard deviation is given by

$$n^{1/2} \frac{B_n}{S_n} \approx n^{1/2} \left[ K \left( \frac{n}{k_n} \right)^{2/\alpha-1} \right]^{-1/2} \left( \frac{n}{k_n} \right)^{1/\alpha-1} \approx K^{-1/2} k_n^{1/2},$$

which tends to infinity as  $n \rightarrow \infty$ . This implies that the bias  $B_n$  dominates the distribution of  $\widehat{\Psi}_n^*$  asymptotically. Consequently, it is vital to correct for this bias. In the next section, we consider a bias-corrected version of the tail-trimmed estimator.

## 2.3 Bias-Corrected Tail-Trimmed Estimator

In this section we introduce the bias-corrected tail-trimmed estimator and show that it is asymptotically normal. To that end, we need to restrict the tails of the importance sampling estimator to satisfy Assumption 1 instead of Assumption 1'—the latter is needed to establish the convergence rate of  $\widehat{\alpha}$ .

It follows from (6) that a natural estimator for the bias is:

$$\widehat{B}_n = \frac{\widehat{\alpha}}{\widehat{\alpha} - 1} \frac{k_n}{n} W_{(k_n)}, \tag{7}$$



where we use  $W_{(k_n)}$  as an approximation of  $l_n$ . Then, the bias-corrected estimator is given by

$$\widehat{\Psi}_n^{(b)} = \widehat{\Psi}_n^* + \widehat{B}_n. \quad (8)$$

To derive a central limit theorem for  $\widehat{\Psi}_n^{(b)}$ , we impose the following technical condition on  $k_n$ :

**Assumption 4** *Let  $\{k_n\}$  be a sequence of positive integers such that  $k_n \rightarrow \infty$  and  $k_n/m_n \rightarrow 0$  as  $n \rightarrow \infty$ .*

Recall that the sequence  $\{m_n\}$  is used to estimate  $\alpha$  in (3). Under Assumption 4,  $m_n$  diverges faster than  $k_n$  so that the first step estimation of  $\alpha$  does not affect the asymptotics of the bias-corrected estimator  $\widehat{\Psi}_n^{(b)}$ . This can be seen from Theorem 2.1, where the convergence rate of  $\widehat{\alpha}$  is  $m_n^{1/2}$ . In the case where we choose  $k_n$  to be the same order of  $m_n$ , the asymptotic distribution of  $\widehat{\Psi}_n^{(b)}$  remains normal, but the asymptotic variance has a complicated expression. For more details, see Peng (2001).

Under Assumption 4, the estimation of  $\alpha$  does not affect the asymptotics of  $\widehat{\Psi}_n^{(b)}$ . However, the use of  $W_{(k_n)}$  in  $\widehat{B}_n$  introduces a term in the influence function of  $\widehat{\Psi}_n^{(b)}$ , which affects the asymptotic variance. We define

$$\begin{aligned} \Upsilon_{n,t} &= \left( W_{n,i}^* - \mathbb{E}W_{n,i}^*, \frac{n^{1/2}}{k_n^{1/2}} [I(W_i > l_n) - \mathbb{P}(W_i > l_n)] \right)', \\ \mathbf{T}_n &= \left( 1, -\frac{1}{\alpha - 1} \frac{k_n^{1/2}}{n^{1/2}} l_n \right)'. \end{aligned}$$

Then, the asymptotic variance of the bias-corrected estimator is given by

$$V_n = \mathbf{T}_n' \boldsymbol{\Omega}_n \mathbf{T}_n,$$

where  $\boldsymbol{\Omega}_n = \mathbb{E} [\Upsilon_{n,t} \Upsilon_{n,t}']$ .

Finally, we state the central limit theorem for  $\widehat{\Psi}_n^{(b)}$  below. The proof is given in Appendix A.

**Theorem 2.3** *Suppose Assumptions 1–3 and 4 hold, then we have*

$$n^{1/2} V_n^{-1} \left( \widehat{\Psi}_n^{(b)} - \Psi \right) \xrightarrow{d} \mathcal{N}(0, 1).$$

Here we discuss the choice of the tuning parameters  $m_n$  and  $k_n$ . Since in our context  $n$  is the simulation size which the user controls, the exact choice of  $m_n$  and  $k_n$  is less important. Our baseline recommendation is to set  $m_n = \sqrt{n} \log n$  and  $k_n = \sqrt{n}$ . This choice then satisfies Assumption 2 and Assumption 4, and we can establish Theorem 2.3. This seems to work well in the Monte Carlo study and it gives reasonable results in our empirical illustration.

Below we discuss other alternatives. In general, the optimal—in the sense of minimizing root mean square errors of  $\hat{\alpha}$ —data-driven procedure of choosing  $m_n$  is difficult. This is because the optimal choice depends critically on the second order tail behavior of  $W_i$  (the unknown parameter  $\beta$  in our case), which is hard to verify in practice.

In one strand of the literature, Hall (1990), Gomes and Oliveira (2001) and Danielsson et al. (2001) suggest bootstrap methods by further imposing certain strong conditions on the tails. In another strand of literature, Resnick (1997) and Resnick and Starica (1997) propose graphical tools—the Hill plot. Recently, Hill (2013) suggests similar methods for the weakly dependent case. The bootstrap methods are computational intensive and require stronger conditions on the tails, while the Hill plot is easy to construct but is more *ad hoc*. However, both methods are often used in the literature (e.g., McNeil, 1997; Drees, de Haan and Resnick, 2000; Danielsson et al., 2001).

Here we suggest an empirical rule to choose  $m_n$  based on Daouia, Florens and Simar (2010). More specifically, we locate a stable region in the alternative Hill plot of  $(u, \hat{\alpha}_u^{-1})$ ,  $0 < u < 1$ , and

$$\hat{\alpha}_u^{-1} = ([n^u])^{-1} \sum_{t=1}^{[n^u]} (\log W_{(t)} - \log W_{(n^u)}),$$

where  $[x]$  returns the smallest integer no less than  $x$ .

This alternative Hill plot is defined in Resnick and Starica (1997) and has nice properties and advantages over the original Hill plot; see Resnick and Starica (1997) for details. To find the stable region, we calculate a sequence of estimates of  $\alpha$ , namely,  $\{\hat{\alpha}_{1/L}, \hat{\alpha}_{2/L}, \dots, \hat{\alpha}_1\}$ .

Then, to measure whether the sequence of estimates has become stable, we compute the standard deviation of a rolling-window of estimates as follows:

$$\text{sd}_i = \sqrt{\text{Var} \left( \hat{\alpha}_{i/L}, \dots, \hat{\alpha}_{(i+(0.05n)^{0.5})/L} \right)}, \quad i = 1, \dots, L - \sqrt{0.05n}.$$

We then locate the “least volatile region” as the index  $i$  such that the standard deviation attains the minimum, i.e.,  $\hat{i}_{\min} = \operatorname{argmin}_i \{\operatorname{sd}_i\}$ . Finally, we set  $m_n = \lceil n^{\hat{i}_{\min}/L} \rceil$ .

For the choice of  $k_n$ , we suggest taking  $k_n = m_n / \log n$ . The reason is twofold. First, this choice is in line with Assumption 4. Second, a larger  $k_n$  implies smaller  $S_n$  and  $V_n$ , which in turn suggests a faster convergence rate of  $\hat{\Psi}_n^*$  and  $\hat{\Psi}_n^{(b)}$ .

In the Monte Carlo experiments next section, we present results based on the baseline recommendation of setting  $m_n = \sqrt{n} \log n$  and  $k_n = \sqrt{n}$ . We also perform a robustness check of the Monte Carlo results using the empirical rule discussed above, which are reported in [Appendix B](#). It turns out that the results are not sensitive to how  $m_n$  and  $k_n$  are chosen.

### 3 Monte Carlo Experiments

In this section we investigate the properties of the proposed bias-corrected tail-trimmed estimator via a series of Monte Carlo experiments. Recall that the goal is to estimate the integral  $\Psi = \int_{\mathcal{X}} H(x) f(x) dx$  using importance sampling. Following [Koopman, Shephard and Creal \(2009\)](#), we assume that  $f(x)$  is a Gaussian density and approximate it using various Gaussian densities with thinner tails.

More specifically, we set  $f(x)$  to be  $\mathcal{N}(0, 1)$  and take  $H(x) = 1$ . It is easy to see that  $\Psi = \int_{-\infty}^{\infty} H(x) f(x) dx = 1$ . The importance sampling density  $g(x)$  is chosen to be  $\mathcal{N}(0, (1 + \epsilon)^{-1})$  with  $\epsilon > 0$ . Hence, the tails of  $g(x)$  are thinner than those of the original density  $f(x)$ . When  $\epsilon$  is sufficiently large, the variance of the importance sampling estimator becomes infinite. To show that, first note that the importance weight  $W(x)$  is given by

$$W(x) = \frac{f(x)H(x)}{g(x)} = \frac{1}{\sqrt{1 + \epsilon}} e^{\frac{\epsilon}{2}x^2}.$$

It is easy to verify that  $\mathbb{E}_g[W(X)] = 1$ . Next, we compute the second moment of  $W(x)$ :

$$\begin{aligned} \mathbb{E}_g[W(X)^2] &= \int_{-\infty}^{\infty} \frac{1}{1 + \epsilon} e^{\epsilon x^2} \times \sqrt{\frac{1 + \epsilon}{2\pi}} e^{-\frac{1 + \epsilon}{2}x^2} dx \\ &= \frac{1}{\sqrt{2\pi(1 + \epsilon)}} \int_{-\infty}^{\infty} e^{-\frac{1 - \epsilon}{2}x^2} dx. \end{aligned}$$

It implies that  $\mathbb{E}_g[W(X)^2] < \infty$  only if  $\epsilon < 1$ . When  $\epsilon \geq 1$ , the variance of the importance

sampling estimator is infinite.

We consider three values of  $\epsilon$ : 0.5, 1 and 3. Hence, only in the first case does the unmodified importance sampling estimator  $\widehat{\Psi}_n$  have a finite variance. In the baseline case we set the tuning parameters for the bias-corrected tail-trimmed estimator as  $m_n = \sqrt{n} \log n$  and  $k_n = \sqrt{n}$ . These tuning parameters satisfy Assumptions 2 and 4. We also perform a robustness check of these Monte Carlo results by using an empirical rule based on the alternative Hill plot to choose  $m_n$  and  $k_n$  as discussed in Section 2.3. The results are reported in Appendix B.

Figures 1-3 plot the sampling distributions of the importance sampling estimator  $\widehat{\Psi}_n$  in (1) and the bias-corrected tail-trimmed estimator  $\widehat{\Psi}_n^{(b)}$  in (8) under different settings. Each data point in the histograms consists of an estimate using a simulation size of  $n = 10000$ , and we use a total of 10000 independent estimates.

It is clear from Figure 1 that when the variance of  $\widehat{\Psi}_n$  is finite and the central limit theorem applies, the sampling distributions of both estimators are approximately Gaussian and centered around the true value of 1. In fact, Table 1 below shows that the quantiles of the two distributions are essentially identical.

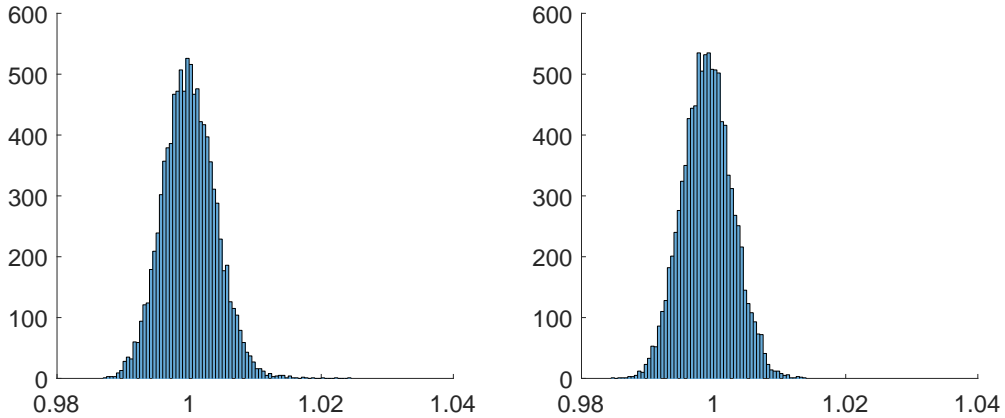


Figure 1: Sampling distributions of the importance sampling estimator  $\widehat{\Psi}_n$  (left panel) and the proposed bias-corrected tail-trimmed estimator  $\widehat{\Psi}_n^{(b)}$  (right panel) with a simulation size of  $n = 10000$ ;  $\epsilon = 0.5$ .

Figure 2 depicts the results for  $\epsilon = 1$ , when the variance of  $\widehat{\Psi}_n$  is infinite and the central limit theorem fails to apply. The sampling distribution of  $\widehat{\Psi}_n$  is noticeably more skewed with many large estimates, despite a substantial simulation size of  $n = 10000$ . In contrast, the sampling distribution of the tail-trimmed estimator  $\widehat{\Psi}_n^{(b)}$  remains approximately

Gaussian and is centered around the true value of 1. This is of course not surprising—the right panel of Figure 2 is simply an empirical verification of the central limit theorem proved in Theorem 2.3.

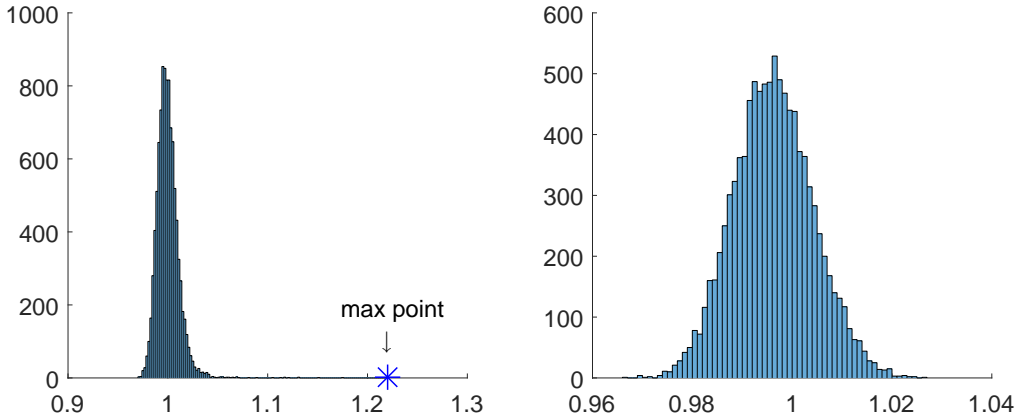


Figure 2: Sampling distributions of the importance sampling estimator  $\hat{\Psi}_n$  (left panel) and the proposed bias-corrected tail-trimmed estimator  $\hat{\Psi}_n^{(b)}$  (right panel) with a simulation size of  $n = 10000$ ;  $\epsilon = 1$ .

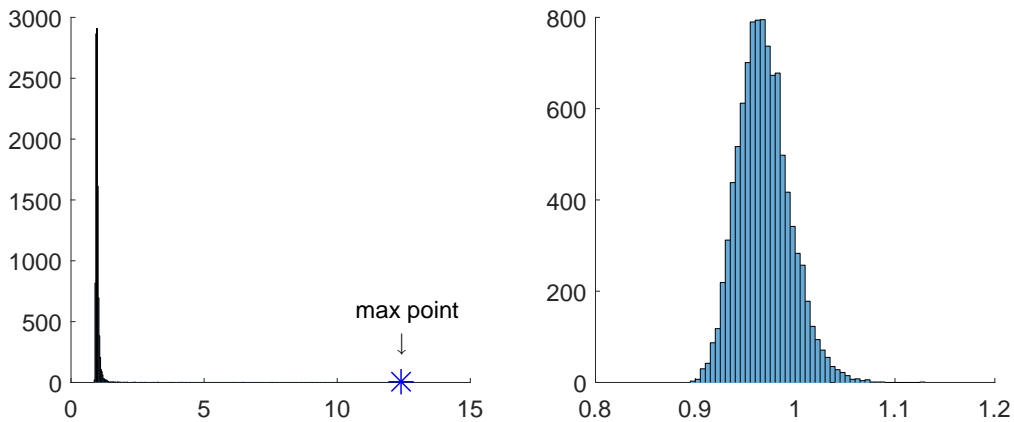


Figure 3: Sampling distributions of the importance sampling estimator  $\hat{\Psi}_n$  (left panel) and the proposed bias-corrected tail-trimmed estimator  $\hat{\Psi}_n^{(b)}$  (right panel) with a simulation size of  $n = 10000$ ;  $\epsilon = 3$ .

When  $\epsilon = 3$ , the tails of the importance sampling density  $g(x)$  are substantially thinner than those of  $f(x)$ . Hence, it is of no surprise that the unmodified importance sampling estimator  $\hat{\Psi}_n$  performs very poorly, with the largest estimate being over 12 times larger than the true value. In fact, Table 1 shows that the 99th percentile of the estimates is 1.365. In other words, 1% of the estimates are over 36.5% larger than the true value.

In contrast, the proposed tail-trimmed estimator  $\widehat{\Psi}_n^{(b)}$  performs reasonably well even in this severe setting. Although there seems to be a small finite-sample bias—the median estimate is 0.967 compared to the true value of 1—its sampling distribution remains approximately Gaussian and the mass of the density is concentrated between 0.9 and 1.05.

Table 1 also presents the quantiles of the two sampling distributions when the simulation size is increased to  $n = 100000$ . The basic conclusions remain the same: when the variance of  $\widehat{\Psi}_n$  is finite, the properties of both estimators are essentially identical. When the variance of  $\widehat{\Psi}_n$  is infinite, its sampling distribution can be severely skewed with many extremely large estimates, even when the simulation size is very large. On the other hand, the proposed tail-trimmed estimator  $\widehat{\Psi}_n^{(b)}$  performs reasonably well with an approximate Gaussian sampling distribution.

Table 1: Quantiles of the sampling distributions of the importance sampling estimator  $\widehat{\Psi}_n$  and the proposed bias-corrected tail-trimmed estimator  $\widehat{\Psi}_n^{(b)}$ .

	$n = 10000$					$n = 100000$				
$\epsilon = 0.5$	min	1%	50%	99%	max	min	1%	50%	99%	max
$\widehat{\Psi}_n$	0.987	0.991	1.000	1.010	1.024	0.995	0.997	1.000	1.003	1.006
$\widehat{\Psi}_n^{(b)}$	0.985	0.991	0.999	1.008	1.014	0.995	0.997	1.000	1.003	1.004
$\epsilon = 1$	min	1%	50%	99%	max	min	1%	50%	99%	max
$\widehat{\Psi}_n$	0.971	0.980	0.999	1.033	1.220	0.989	0.993	1.000	1.010	1.838
$\widehat{\Psi}_n^{(b)}$	0.967	0.979	0.996	1.015	1.027	0.989	0.992	0.998	1.005	1.010
$\epsilon = 3$	min	1%	50%	99%	max	min	1%	50%	99%	max
$\widehat{\Psi}_n$	0.880	0.910	0.972	1.365	12.42	0.939	0.951	0.986	1.196	34.23
$\widehat{\Psi}_n^{(b)}$	0.898	0.917	0.967	1.038	1.127	0.948	0.959	0.982	1.009	1.026

Next, we study the empirical convergence rates of the two estimators under the settings  $\epsilon = 1$  and  $\epsilon = 3$ . Since the mean squared error for  $\widehat{\Psi}_n$  does not exist for these settings, we instead consider the interquartile range, defined as the difference between the 75th percentile and the 25th percentile. For a normal random variable with variance  $\sigma^2/n$ , its interquartile range is  $2z_{0.75}\sqrt{\sigma^2/n}$ , where  $z_{0.75}$  is the 75th percentile of the standard normal distribution. Theorem 2.3 indicates that the convergence rate of the bias-corrected tail-trimmed estimator  $\widehat{\Psi}_n^{(b)}$  depends of the tail parameter  $\alpha$  and is typically slower than the usual root- $n$  rate.

Figure 4 plots the interquartile range (in log) against the simulation size  $n$  (in log) for both estimators. In the case  $\epsilon = 1$ , the slopes corresponding to  $\widehat{\Psi}_n$  and  $\widehat{\Psi}_n^{(b)}$  are respectively

−0.439 and −0.455—the convergence rate of the proposed estimator is slightly faster. When  $\epsilon = 3$ , however, the difference becomes much larger—the empirical convergence rates of  $\widehat{\Psi}_n$  and  $\widehat{\Psi}_n^{(b)}$  become  $n^{0.274}$  and  $n^{0.359}$ , respectively. These values imply that if we want to halve the width of the interquartile range of  $\widehat{\Psi}_n^{(b)}$ , we need to increase the simulation size by 5.5 times (compared to 4 times in usual settings). For  $\widehat{\Psi}_n$ , however, we need to increase the simulation size by 7.3 times.

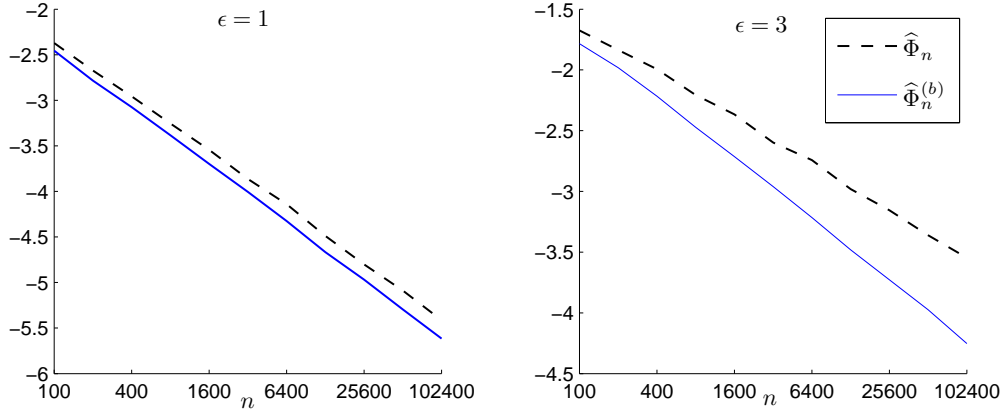


Figure 4: The interquartile range (in log) against the simulation size  $n$  (in log) for the importance sampling estimator  $\widehat{\Psi}_n$  and the proposed bias-corrected tail-trimmed estimator  $\widehat{\Psi}_n^{(b)}$ .

## 4 An Illustration

In this section we illustrate the performance of the bias-corrected tail-trimmed estimator using an empirical application in which we fit a stochastic volatility model using daily returns on the Standard & Poor 500 (S&P 500) index. Stochastic volatility models are widely used to model financial returns. Notable early examples include Danielsson (1994), Durbin and Koopman (1997) and Shephard and Pitt (1997).

The daily returns on S&P 500 index are from September 2012 to August 2016, which are sourced from the Federal Reserve Bank of St. Louis economic database:

<https://fred.stlouisfed.org/series/SP500>.

Specifically, let  $P_t$  denote the S&P 500 index on day  $t$ . We compute the return on day  $t$

as  $y_t = 100 \log(P_t/P_{t-1})$ . The standard stochastic volatility model is given as

$$y_t = \mu + \epsilon_t^y, \quad \epsilon_t^y \sim \mathcal{N}(0, e^{h_t}), \quad (9)$$

$$h_t = \mu_h + \phi_h(h_{t-1} - \mu_h) + \epsilon_t^h, \quad \epsilon_t^h \sim \mathcal{N}(0, \omega_h^2), \quad (10)$$

where the initial log volatility is initialized by  $h_1 \sim \mathcal{N}(\mu_h, \omega_h^2/(1 - \phi_h^2))$ .

We estimate this model using Markov chain Monte Carlo methods (e.g., [Kim, Shepherd and Chib, 1998](#)). In particular, we obtain the posterior mean of the parameter vector  $(\mu, \mu_h, \phi_h, \omega_h^2)$ . We then evaluate the likelihood at this posterior mean by integrating out the log volatility  $\mathbf{h} = (h_1, \dots, h_T)'$  using importance sampling.

The importance sampling density is obtained by approximating the posterior distribution of  $\mathbf{h}$  given the parameters using a Gaussian distribution as in [Durbin and Koopman \(1997\)](#). But instead of using the Kalman filter to obtain the importance sampling density, band matrix routines as discussed in [Chan and Grant \(2016\)](#) and [Chan \(2017\)](#) are used. Furthermore, independent draws from this Gaussian importance sampling density are obtained using the precision sampler in [Chan and Jeliazkov \(2009\)](#), which is more efficient than Kalman filter-based algorithms.

Figure 5 plots the sampling distributions of the importance sampling estimator  $\log \widehat{\Psi}_n$  and the bias-corrected tail-trimmed estimator  $\log \widehat{\Psi}_n^{(b)}$ . Each data point in the histograms consists of an estimate using a simulation size of  $n = 100000$ .

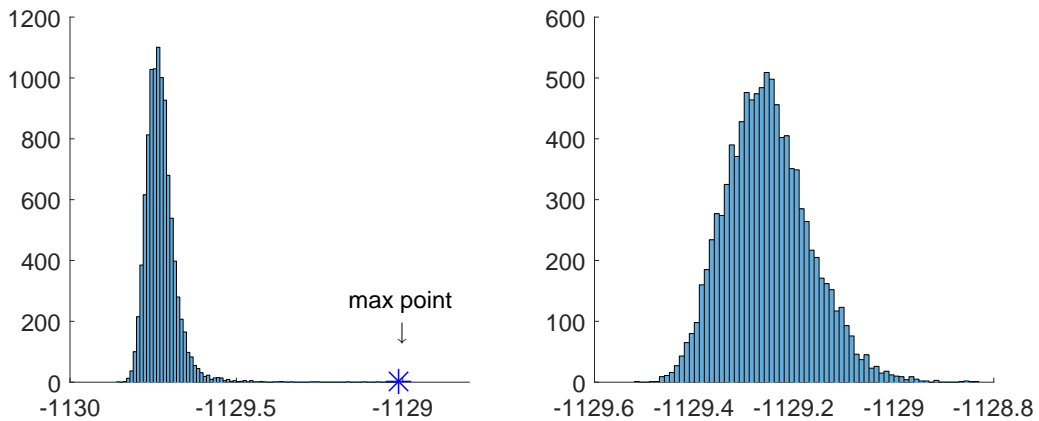


Figure 5: Sampling distribution of the importance sampling estimator  $\log \widehat{\Psi}_n$  (left panel) and the proposed bias-corrected tail-trimmed estimator  $\log \widehat{\Psi}_n^{(b)}$  (right panel) with a simulation size of  $n = 100000$ .



As is evident from the figure, the sampling distribution of  $\log \widehat{\Psi}_n$  is substantially more skewed even though the simulation size is  $n = 100000$ . In contrast, the sampling distribution of the proposed estimator is less dispersed and remains approximately Gaussian.

Next, we plot in Figure 6 the sampling distribution of the ratio of the two estimators  $\widehat{\Psi}_n^{(b)}/\widehat{\Psi}_n$ . Here the same importance sampling weights are used to compute the two estimates. Again each data point in the histogram consists of an estimate using a simulation size of  $n = 100000$ .

The distribution is centered at around 1.6 with virtually no mass near unity, indicating that the  $\widehat{\Psi}_n$  estimates tend to be substantially smaller than those of  $\widehat{\Psi}_n^{(b)}$ . This suggests that the original importance sampling estimator might underestimate the likelihood value  $\Psi$ .

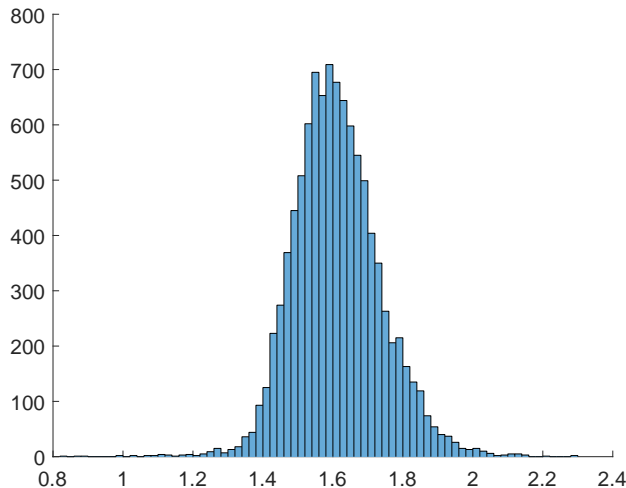


Figure 6: Sampling distribution of the ratio  $\widehat{\Psi}_n^{(b)}/\widehat{\Psi}_n$  with a simulation size of  $n = 100000$ .

## 5 Concluding Remarks and Future Research

We have proposed a way forward when an importance sampling estimator has an infinite variance. Specifically, we have shown how one could modify the original infinite-variance estimator to obtain a consistent estimator that is asymptotically normal. The proposed estimator performed well in the Monte Carlo study as well as the empirical illustration.

In this paper we have only considered the case where independent samples are used to compute the importance sampling estimate. In many Bayesian applications, MCMC

draws are used instead—and by construction these draws are correlated. Hence, for future work it would be worthwhile to first develop a similar test as [Koopman, Shephard and Creal \(2009\)](#) for the weakly dependent case. In addition, it would also be fruitful to consider asymptotic trimming in such a setting.

## Appendix A Main Proofs

Below we apply the results in Hill (2013) to prove Theorem 2.2 and Theorem 2.3.

**Proof of Theorem 2.2:** We apply the same reasoning and derivations of Theorem 1.2 in Hill (2013). Using the notations in his proof, replace  $\alpha, \Psi, B_n$  with  $\kappa, S, \alpha^{-1}\mathbb{E}[y_t I(y_t < -l_n)]$ , respectively. Also note that our standard deviation is denoted by  $S_n$  while his is  $\alpha^{-1}\mathcal{S}_n$ .

Since Hill (2013) deals with the more general dependent case—whereas we focus on the independent case— Assumption N in Hill (2013) is automatically satisfied. Assumption D and Assumption T required in Hill (2013) can be verified by using our Assumption 3 and Assumption 1', respectively. *Q.E.D.*

**Proof of Theorem 2.3:** We again apply Theorem 2.1 in Hill (2013) by replacing  $\alpha, \Psi, B_n$  with  $\kappa, S, \alpha^{-1}\mathbb{E}[y_t I(y_t < -l_n)]$ , respectively. Here our standard deviation is denoted by  $V_n$ , whereas his is  $\alpha^{-1}\mathcal{V}_n$ . By the same reasoning as above, Assumption N' in Hill (2013) is automatically satisfied. Assumption D, Assumption T' and (5)-(6) required in Hill (2013) can be verified by using our Assumption 3, Assumption 1, Assumption 4, and Assumption 2, respectively. *Q.E.D.*

## Appendix B Additional Results: Robustness Checks

In this appendix we redo the Monte Carlo study in Section 3 by using an empirical rule based on the alternative Hill plot to choose  $m_n$  and  $k_n$  discussed in Section 2.3. The sampling distributions of  $\widehat{\Psi}_n$  and  $\widehat{\Psi}_n^{(b)}$  for the cases  $\epsilon = 0.5$ ,  $\epsilon = 1$  and  $\epsilon = 3$  are reported below. The results are very similar to those reported in Section 3 using the baseline choice of  $m_n$  and  $k_n$ .

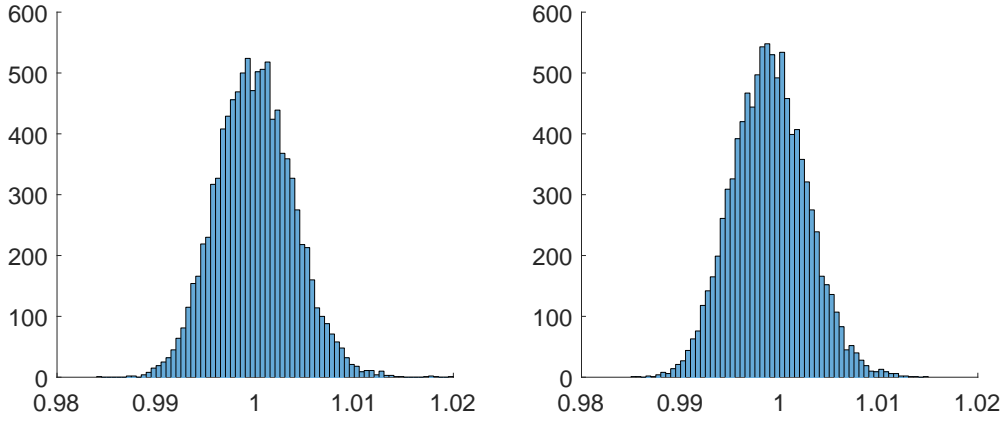


Figure 7: Sampling distributions of the importance sampling estimator  $\widehat{\Psi}_n$  (left panel) and the proposed bias-corrected tail-trimmed estimator  $\widehat{\Psi}_n^{(b)}$  (right panel) with a simulation size of  $n = 10000$ ;  $\epsilon = 0.5$ .

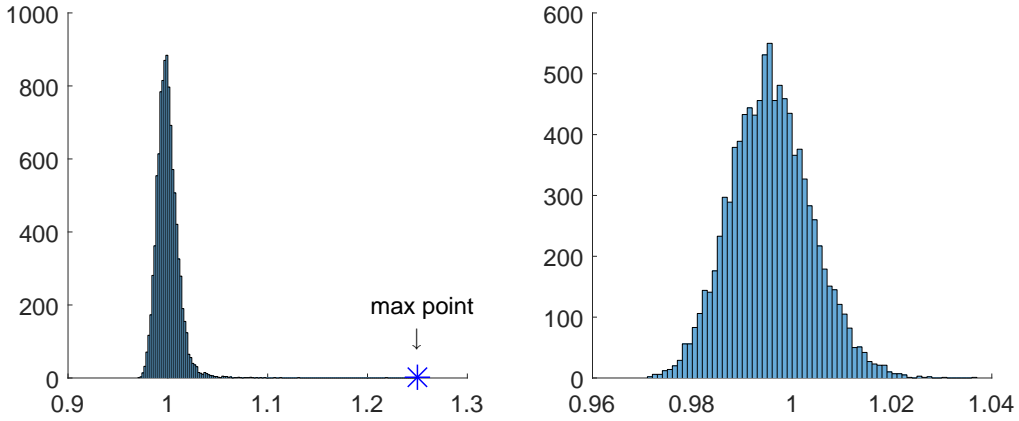


Figure 8: Sampling distributions of the importance sampling estimator  $\widehat{\Psi}_n$  (left panel) and the proposed bias-corrected tail-trimmed estimator  $\widehat{\Psi}_n^{(b)}$  (right panel) with a simulation size of  $n = 10000$ ;  $\epsilon = 1$ .

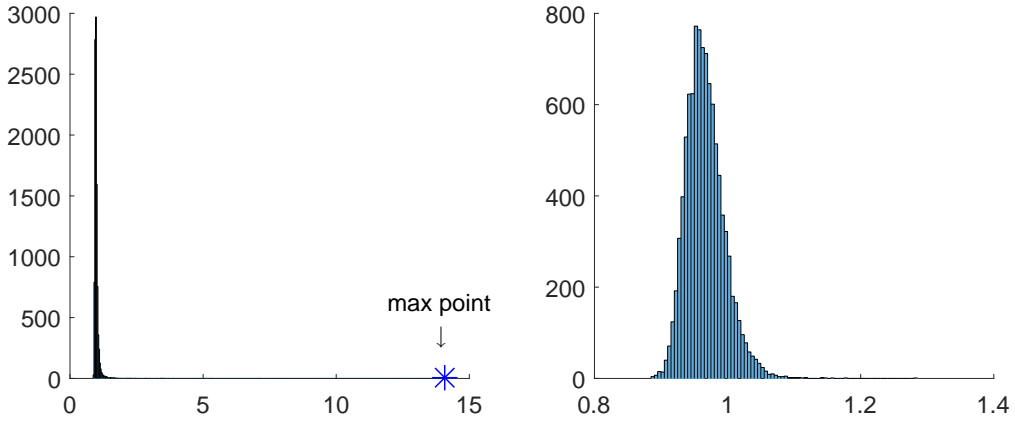


Figure 9: Sampling distributions of the importance sampling estimator  $\hat{\Psi}_n$  (left panel) and the proposed bias-corrected tail-trimmed estimator  $\hat{\Psi}_n^{(b)}$  (right panel) with a simulation size of  $n = 10000$ ;  $\epsilon = 3$ .

Table 2: Quantiles of the sampling distributions of the importance sampling estimator  $\hat{\Psi}_n$  and the proposed bias-corrected tail-trimmed estimator  $\hat{\Psi}_n^{(b)}$ .

	$n = 10000$					$n = 100000$				
$\epsilon = 0.5$	min	1%	50%	99%	max	min	1%	50%	99%	max
$\hat{\Psi}_n$	0.984	0.991	1.000	1.009	1.020	0.996	0.997	1.000	1.003	1.007
$\hat{\Psi}_n^{(b)}$	0.985	0.991	0.999	1.008	1.015	0.995	0.997	1.000	1.003	1.005
$\epsilon = 1$	min	1%	50%	99%	max	min	1%	50%	99%	max
$\hat{\Psi}_n$	0.972	0.979	0.999	1.036	1.250	0.989	0.993	1.000	1.011	1.076
$\hat{\Psi}_n^{(b)}$	0.971	0.978	0.996	1.016	1.032	0.990	0.993	0.999	1.007	1.012
$\epsilon = 3$	min	1%	50%	99%	max	min	1%	50%	99%	max
$\hat{\Psi}_n$	0.883	0.908	0.973	1.358	14.05	0.935	0.952	0.985	1.207	13.23
$\hat{\Psi}_n^{(b)}$	0.885	0.912	0.964	1.049	1.280	0.940	0.961	0.989	1.037	1.103

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